Two simple formulas for climbing fall forces for static and dynamic belays

July 2022

Ulrich Leuthäusser

There are two simple formulas for estimating the size of the forces during a climbing fall. The first is the well-known impact force formula for a fix-point belay [1-4]. That is, the rope brake is directly attached to the belay and in case of a fall of the climber it is more or less motionless. In popular presentations this impact force formula is still used to calculate the appearing forces during a climbing fall. But this formula does not take into account that the common belay method in sport climbing is a static rope brake attached to the belayer's harness, but the belayer can move freely.
In the following, this new impact force formula [5] which has to be applied in the latter case of a moving belayer is derived and compared with the formula for the fix-point situation. Furthermore, these formulas are applied to estimate the size of the forces appearing in a climbing fall using only climbing rope data of the manufacturers which are accessible for everyone.

Let's at first recapitulate the static impact force formula with a fix-point belay. To keep it simple, we only use the principle of conservation of energy to avoid the equations of motions and we assume that the rope obeys Hooke' law. The situation is presented in Fig. 1 with a fixed belay at B (x_0 always zero).

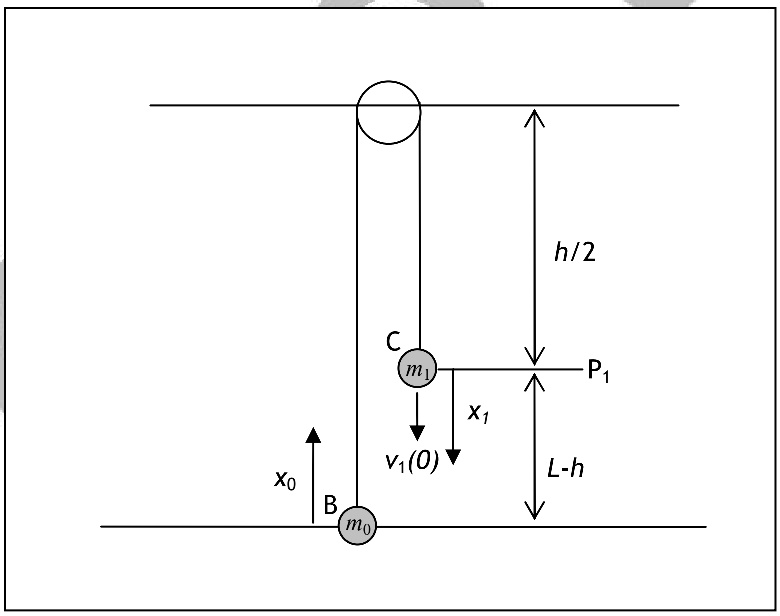

Figure1. A belayer B of mass m_0 is attached to a climber C with mass m_1 who fell a distance h and has the velocity $v_1(0) = \sqrt{2gh}$ at P_1 where the rope begins to stretch. The motion of B is described by the coordinate x_0, C by the coordinate x_1.

At any time the initial kinetic energy K_0 of the falling climber must be equal to the sum of his remaining kinetic energy K_1, his lower gravitational potential energy U_g and the elastic energy U_s of the stretched rope. Hooke's law states that the restoring spring force F is proportional to the elongation x_1 of the rope. Such a linear force-elongation relationship leads to a quadratic potential spring energy (force times displacement!) stored in the spring. Setting up the balance equation for the conservation of energy, one obtains:

$$K_0 = K_1 + U_g + U_s$$

and explicitly

$$\frac{1}{2} m_1 v_1(0)^2 = \frac{1}{2} m_1 v_1(t)^2 - m_1 g x_1(t) + \frac{1}{2} k x_1(t)^2 \tag{1}$$

with

m_1 as the mass of the climber,
$v_1(0)$ as the initial velocity of the climber at the time when the rope begins to stretch after falling a height h (see Figure 1),
$v_1(t)$ as the velocity of the climber during the fall after the rope began to stretch,
$x_1(t)$ as the elongation of the rope,
g as the gravitational acceleration constant and
k as the spring constant.

It is important to realize that the spring constant k is a function of the paid-out rope length L as well as a function of the cross sectional area A of the rope. This is easy to understand, because a longer rope can be stretched more easily than a shorter one resulting in a smaller spring constant. On the other hand it is more difficult to stretch a thicker rope. Therefore it is useful to write k in terms of the more fundamental elasticity modulus E which is only dependent on the rope material which leads to

$$k = E\frac{A}{L} \ . \tag{2}$$

Expressing $v_1(0)$ in terms of h, one obtains $v_1(0) = \sqrt{2gh}$. h is often replaced by the fall factor f=h/L. The maximum force F_{max} occurs at maximum elongation $x_{max}=F_{max}/k$ at the turning point of the rope. At that point, the velocity of the climber is zero, that is $v_1=0$. Using all this, equation (1) changes to

$$m_1 g f E A = -m_1 g F_{\max} + \frac{1}{2} F_{\max}^2 \tag{3}$$

Solving for F_{max}, one gets the well-known impact force formula

$$F_{\max} = m_1 g + \sqrt{2 m_1 g f E A + (m_1 g)^2} \tag{4}$$

With the available values of the UIAA standard fall ($F_{\max}^{norm}$ between 7.6kN and 9.5kN depending on the type and brand of the rope, f^{norm} =1.77 and m^{norm} = 80kg, one can calculate EA of a rope

$$EA = \frac{F_{\max}^{norm}}{f^{norm}}\left(\frac{F_{\max}^{norm}}{2 g m^{norm}} - 1\right). \tag{5}$$

which varies between 16.1kN and 26.5kN. Note that EA is the dynamic elastic modulus which is larger (that is stiffer) than the static modulus. If one uses the above values of EA, the static elongation gm^{norm}/EA lies between 3% and 5%, much lower than the observed 6-8%. That's why more complicated viscoelastic rope models like the SLS model [6] are applied. Introducing more parameters, however, does not explain the physics of this difference between the dynamic and static elastic modulus.

For larger falls, i.e. for $2fEA >> m_1 g$, equation (4) is approximately given by

$$F_{\max} >\approx F_{\max}^{norm} \sqrt{\frac{f}{f^{norm}} \frac{m_1}{m^{norm}}} \, .$$

This is good news. F_{max} increases only with the square root of f. But let's plug in some numbers. For a typical sport climbing fall where the feet of the climbers are close to the last bolt, one has a fall height of about 2 meters. For a paid-out rope length of 10m the fall factor is f=0.2 leading to $\sqrt{f/f^{norm}} \approx 0.33$. For a rope length of 5m one has $\sqrt{f/f^{norm}} \approx 0.5$.

Assuming that m_1 is about equal to m_{norm}, the maximum force in a typical fall lies between 1/3 and 1/2 of the UIAA norm fall, that is between 2.8kN and 4.2kN . This is still a large number of 4-6 climber weights. The force on the last bolt or the last protection point which has to hold the fall is about 2 times the impact force. 5.6kN - 8.4kN is no problem for a modern bolt. For a small BD Camalot, however, it is. In a documented long climbing fall in a multi-pitch route in the Dolomites, a chain reaction of several failing protection points happened. The fall had a fall factor f about 0.4. The first failing protection point was a Camalot 0.3 which was deformed and pulled out.

Thus, the forces that appear during relatively small climbing falls should not be underestimated.

Considering now a movable belayer, we start as before with the equation of conservation of energy and include the motion of the belayer

$$\frac{1}{2} m_1 v_1(0)^2 = \frac{1}{2} m_1 v_1(t)^2 - m_1 g x_1(t) + \frac{1}{2} m_0 v_0(t)^2 + m_0 g x_0(t) + \frac{1}{2} k \left(x_1(t) - x_0(t)\right)^2 \tag{6}$$

with

m_0 as the mass of the belayer,
$v_0(t)$ as the velocity of the belayer during the fall,
$x_0(t)$ as the distance of the belayer from the ground with x0(0)=0.
$x_1(t)$ as the distance of the climber from P1 (see Fig. 1).

For an immobile belayer ($v_0(t)$=0 and $x_0(t)$=0), the old equation (1) is regained. Expressing x_0 and x_1 by the relative coordinate $d = x_1 - x_0$ and the center of mass coordinate $x_c = \frac{m_0 x_0 + m_1 x_1}{m_0 + m_1}$ one gets

$$\begin{aligned} x_0 &= x_c - \frac{m_1}{m_0 + m_1} d \\ x_1 &= x_c + \frac{m_0}{m_0 + m_1} d \end{aligned} \tag{7}$$

By means of the equations (7), x_0 and x_1 in equation (6) are replaced by d and x_c. After some calculations one finds

$$\frac{1}{2}m_1 v_1(0)^2 = \frac{1}{2}(m_0 + m_1)v_c(t)^2 + (m_0 - m_1)gx_c(t) - 2gm_r d(t) + \frac{1}{2}m_r v_d(t)^2 + \frac{1}{2}kd(t)^2 \quad (8)$$

where the reduced mass

$$m_r = \frac{m_0 m_1}{m_0 + m_1} \quad (9)$$

and v_d as the velocity of the relative coordinate d have been introduced. d describes the elongation of the rope and thus its restoring force. In equation (8) there is no coupling between x_c and d so that the energy of the center of mass system (the first two terms) is separated from an oscillator with mass m_r with a gravitational constant 2g. Because the center of mass motion is independent of the relative motion for all times, the initial energy of the relative motion is given by $\frac{1}{2}m_r v_d(0)^2 = \frac{1}{2}m_r v_1(0)^2$. The energy conservation equation for the relative motion is therefore

$$\frac{1}{2}m_r v_1(0)^2 = -2gm_r d(t) + \frac{1}{2}m_r v_d(t)^2 + \frac{1}{2}kd(t)^2 \quad (10)$$

At the turning point with $v_d(t)=0$ one has

$$\frac{1}{2}m_r v_1(0)^2 = -2gm_r d_{\max} + \frac{1}{2}kd_{\max}^2$$

and with $v_1(0) = \sqrt{2gh}$ and $d_{\max} = \frac{1}{k}\hat{F}_{\max}$ follows

$$m_r gfEA = -2gm_r\hat{F}_{\max} + \frac{1}{2}\hat{F}_{\max}^2 \quad (11)$$

The only difference between the equations (3) and (11) is the replacement of m_1 by the reduced mass m_r and a factor 2 in $2gm_r\hat{F}_{\max}$ because gravity pulls on the climber as well as on the belayer. The solution of equation (11) is

$$\hat{F}_{\max} = 2m_r g + \sqrt{2m_r gfEA + (2m_r g)^2} \quad (12)$$

This equation is not quite correct, because this simple calculation does not exclude that the belayer's position can never be negative (i.e. below the floor). This constraint which prevents this downward motion has been discussed in length in [5] where it is shown that equation (12) is a very good approximation for all reasonable mass ratios m_0/m_1. For larger falls i.e. for $fEA >> 2m_r g$ a simple relationship between $\hat{F}_{\max}$ and $F_{\max}$ results

$$\frac{\hat{F}_{\max}}{F_{\max}} \approx \sqrt{\frac{m_0}{m_0 + m_1}}$$

using $F_{\max} \approx \sqrt{2m_1 gfEA}$ and $\hat{F}_{\max} \approx \sqrt{2m_r gfEA}$. A small m_0 leads to low $\hat{F}_{\max}/F_{\max}$ ratios. The price which has to be paid is a large x_1 with the danger that the climber hits the ground. For equal masses the ratio is $1/\sqrt{2} \approx 0.71$. Thus the force reduction of the movable belayer method is about 30% for large falls. Typically, the position of the belayer is not directly below the climber respectively below the first protection point but has a certain offset with an angle α. In this case, m_0 has to be multiplied with cos α which additionally reduces m_r.

If the belayer jumps off the ground with velocity $v_0(0)$, the difference $v_1(0) - v_0(0)$ enters equation (12) which is changed to

$$\hat{F}_{\max} = 2m_r g + \sqrt{2m_r gfEA\left(1 - \frac{v_0(0)}{v_1(0)}\right)^2 + (2m_r g)^2} \tag{13}$$

Assuming that the movable belayer hits the start time exactly, he can reduce $\hat{F}_{\max}$ by an additional factor $\left(1 - v_0(0)/v_1(0)\right)$. If the belayer jumps with a velocity of $v_0(0)$ ~2.5m/sec from the ground (this corresponds to a height of 30cm of a person jumping upwards) at the beginning of a fall of 2m ($v_1(0)$ ~6.3m/sec) an additional 40% force reduction is gained. All together there is a remaining force of ca. 40% compared to the force for a fixed belay.
Also the opposite sign of $v_0(0)$ is possible: dangerously large forces can appear when the belayer moves quickly backwards during the climber's fall.

In summary, to get an idea of the forces appearing in a climbing fall, two simple formulas are presented which depend on the belay method.
For a fixed- point belay even small falls which happen all the time in sport climbing cause relatively large forces of about one third of the UIAA norm impact force, still several climber masses times g. In multi-pitch routes with small mobile protection devices this can be already a problem.
It has also been shown how the impact force with a movable and active belayer has to be calculated and that this belay method significantly reduces the impact forces.

References

[1] A.Wexler, *The Theory of Belaying*, reprinted from THE AMERICAN ALPINE JOURNAL Vol.VII, 4 (1950)

[2] R.Goldstone, *The Standard Equation For Impact Force* (2006), www.rockclimbing.com/cgi-bin/forum/gforum.cgi?do=post_attachment;postatt_id=746

[3] T.Czermin, P.Dullnig, L.Mathelitsch, W.Schneider, *Bergsteigen und Klettern – was sagt die Physik dazu?*, Physik in unserer Zeit 2 (2001)

[4] U.Leuthäusser, The omnipresent impact force formula for a climbing rope, www.sigmadewe.com/bergsportphysik.html

[5] U.Leuthäusser, The mechanics of a climbing fall with a belayer who can be lifted, Proc IMechE, Part P: J Sports Engineering and Technology (2021) ttps://doi.org/10.1177/1754337121992616

[6] U.Leuthäusser, The physics of a climbing rope under a heavy dynamic load. Proc IMechE, Part P: J Sports Engineering and Technology 231(2): 125–135 (2017)

