About the helplessness of a freely suspended climber
or
the inability to excite a pendulum oscillation on a long rope

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Ulrich Leuthäusser

„Insanity is when doing the same thing over and over again and expecting different results." Albert Einstein

Introduction

It happened in a remote part of the Verdon Canyon. The author abseiled down a steep part of the cliff. The next abseil point according to the guidebook seemed too close to him and so he decided to rappel to the next one, about 40 meters away. The double rope was hanging away from the wall, but not so far that it seemed necessary to get into a swing in order to reach the belay – besides it wasn't that steep. Once on the height of the abseil point, he was about 1.5 meters away from the wall. That's not much and with the help of his legs he was sure that he would reach the wall with a small pendulum swing. From observations of other climbers' helpless attempts and from own fruitless efforts in the climbing garden he was aware that it would be difficult to oscillate with a larger amplitude on the freely suspended rope. The new experience was that almost no oscillation could be excited to reach the necessary several centimeters to the chains of the abseil point.
On a child's swing, however, it is easily possible to set yourself in oscillations from the rest position without being pushed. This raises the question why this was impossible in the described case and why one ended like a "jambon dans le vide" away from the wall. Would a certain way of swinging have led to success, was it just incompetence, or is it in principle not possible to set oneself in a pendulum oscillation?

This question will be answered below.
From the point of view of physics, the climber hanging freely on a rope is a special elastic double pendulum. Although this is conceptually simple, it performs highly complicated movements at larger amplitudes and is a frequently used example of a chaotic system. The equations of motion are set up in the appendix and discussed in more detail in their linear form in the main part. What effect has the elasticity of the rope? Is it possible to stimulate elastic longitudinal oscillations rather than the desired pendulum motion?

Fig.: One of the craziest abseil points in the Verdon Canyon. Without swinging one remains away from the second abseil point by some meters and is still 70 m above the ground. The question is whether a pendulum oscillation of the rope can be excited in order to reach the wall.

A special elastic double pendulum

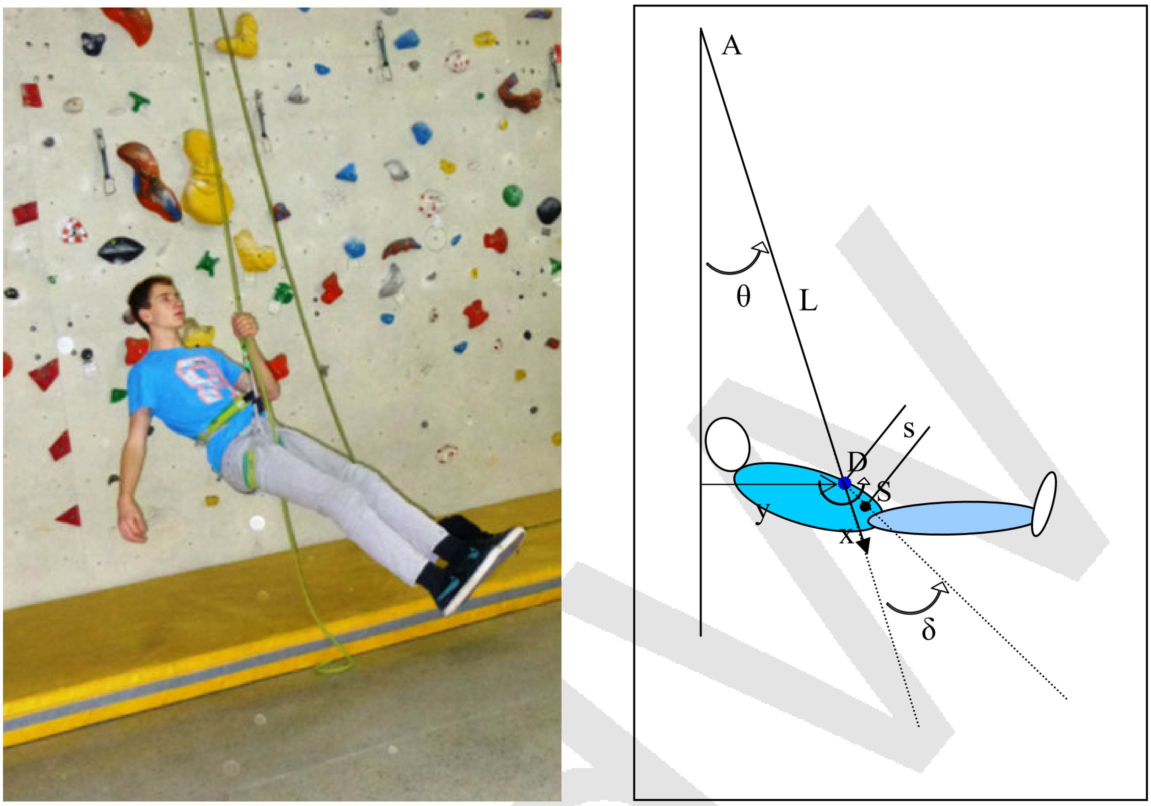

Fig.1a,b: Modeling the freely suspended climber as an elastic double pendulum

The rope represents the first pendulum of the elastic double pendulum. Suspended at its pivot A, it can swing freely (see Fig.1) with the angle θ. It is elastic in longitudinal direction, has an unstretched length L, the spring constant k and is assumed to be massless. At point D, the rope is connected to the second pendulum, the climber K. He has the mass M and the moment of inertia J about an axis through his center-of-mass S and perpendicular to the image plane. K is assumed to be a rigid body, although K moves his hips and lower legs in order to rotate. The rotation around D is described by the angle δ relative to the first pendulum, since the movements of K are independent of θ. The pivot point D generally does not coincide with S and the distance between D and S is s.

A similar double pendulum can be found in the work of W.Case and M.Swanson [1]. They discuss the excitation of a swing and the swinging person is modeled as a rigid barbell. The center-of-mass of the swinging person is above the pivot point, in contrast to the suspension point with a climbing harness. Further work on excitation of a swing are [2], [3] and [4]. [2] examines the standing position on a swing, [3] does without the mathematical apparatus of analytical mechanics and [4] is an experimental work (a swing length L ~ 1.75m is used).

The most elegant method to set up the equations of motion for this mechanical system is the Lagrangian formalism. This requires the Lagrangian function, which results relatively easily from geometric considerations.
These calculations can be found in the appendix, together with the resulting equations of motion for the time evolution of the angle θ and the radial oscillation of the rope with the displacement x. In general terms, these coupled nonlinear differential equations have to

be solved numerically. However, since we are only interested in small angles θ, we can discuss them in a simplified form.

This is obtained from equation (A.3) when also δ is linearized

$$(M(L+s)^2 + J)\ddot{\theta} + Mg(L+s)\theta = -(J + Ms(L+s))\ddot{\delta} - Msg\delta \qquad (1)$$

On the right side of the equation is the driving force that arises from the periodic swinging back and forth of K around the point D at an angle δ. In contrast to the usual problem in mechanics with two degrees of freedom for the double pendulum, where both θ and δ satisfy an equation of motion, there is only one degree of freedom here, because δ is an “external” variable that is controlled by K.
x does not appear in the approximated equation (1), so the motion of θ is completely decoupled from the elastic oscillation x. Since the pendulum length L >> s and the moment of inertia around S $ML^2 >> J$, equation (1) can be further simplified:

$$\ddot{\theta} + \omega_1^2\theta = -\left(\frac{J}{ML^2} + \frac{s}{L}\right)\ddot{\delta} - \frac{sg}{L^2}\delta \qquad (2)$$

This equation describes a harmonic oscillator with the eigenfrequency $\omega_1 = \sqrt{g/L}$ and with a time-dependent driving force on the right side of the equation. An excitation of δ from rest is also possible for s=0, i.e. even if the pivot point is in the center of mass (see Fig.2).

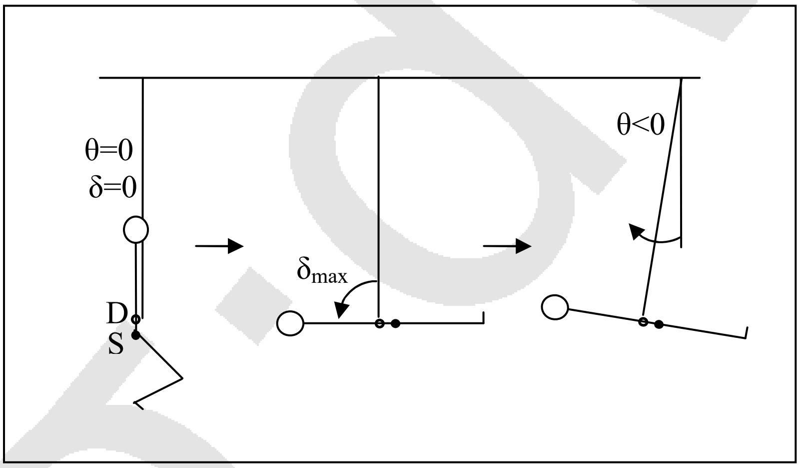

Fig.2: Conservation of angular momentum.
From the starting position $\delta(0)=0$, $\theta(0)=0$ an angular momentum is generated when K is leaning backwards. Initially angular momentum is preserved, i.e. $\left(ML^2\dot{\theta} + J\dot{\delta}\right) = 0$, therefore the pendulum first moves in the other direction.

The pendulum excitation is necessarily periodic and since a negative δ is not possible, because this movement is blocked by the rope, δ varies between zero and a maximum δ_{max}

$$\delta(t) = \frac{\delta_{max}}{2}(1 - \cos(\Omega t)) \qquad (3)$$

The initial δ is $\delta(0)=0$, i.e. one begins in the upright, sitting position from the rest position $\theta(0)=0$. If one choses $\delta\sim\cos(\Omega t)$, then there is no driving force at all at the frequency $\sqrt{sg/(J/M+sL)}$ that is close to the eigenfrequency of K, because the right side of (2) is zero. Thus starting wrongly from the horizontal, lying position ($\delta(0)=90^\circ$, $\theta=0$) then there is no motion at all.

With $\delta(t)$ from equation (3), the solution of equation (2) gives the motion of the angle $\theta(t)$ over time:

$$\theta(t)=\frac{\delta_{\max}}{2}\left[\left(\frac{J}{ML^2}+\frac{s}{L}\right)\Omega^2\left(\frac{\cos(\omega t)-\cos(\Omega t)}{\omega^2-\Omega^2}\right)-\frac{sg}{L^2\omega^2}\left(1+\frac{\Omega^2\cos(\omega t)-\omega^2\cos(\Omega t)}{\omega^2-\Omega^2}\right)\right] \tag{4}$$

At the resonance frequency $\Omega=\omega_1$ this results in

$$\theta(t)=-\frac{J}{2ML^2}\frac{\delta_{\max}}{2}\Omega\sin(\Omega t)\cdot t \tag{5}$$

The amplitude increases linearly with time and is independent of s. The rate $\dot\theta(t)$ at which $\theta(t)$ increases has a strong L dependence $\propto L^{-5/2}$.

A comparison of $\theta(t)$ with the excitation function $\delta(t)$ shows how K has to move in order to excite the pendulum oscillation (see Fig.3).

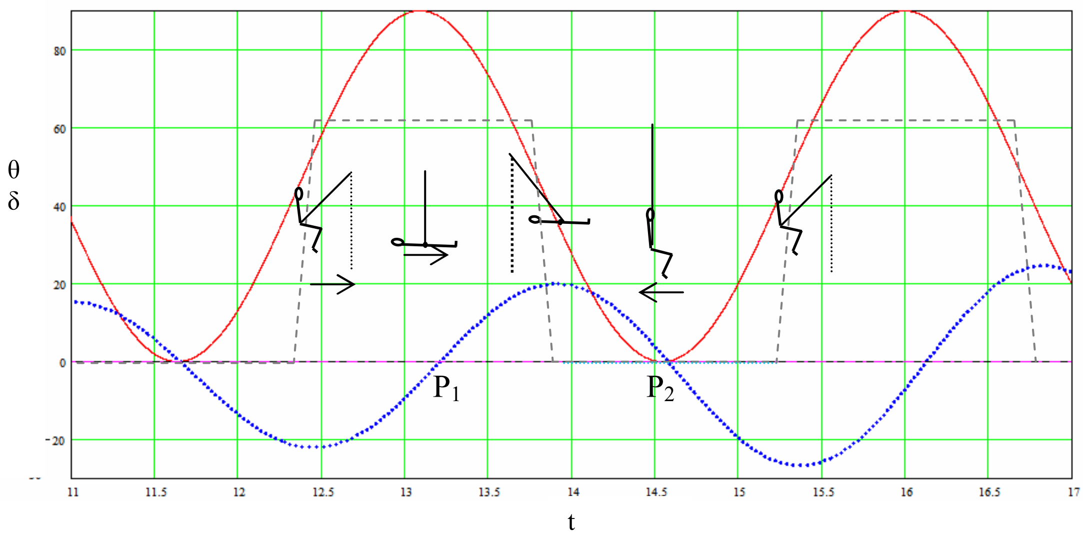

Fig.3: δ is plotted in red as a function of time t, with its maximum δ_{max}=90°corresponding to the "lying" position of K. The blue curve describes $\theta(t)$ in response to the constraint $F(\delta(t))$ in the resonance case. P1 and P2 are the zero crossings of $\theta(t)$. When moving forward at the lowest point P1, K is in a lying, horizontal position. When swinging back, K is in the seated, upright position at point P2. It can be shown that only the reverse swing is able to transfer energy to θ.

A rectangular oscillation, as a 2π-periodic function $\delta(t)=\delta(t+2\pi/\Omega)$, leads to the same $\theta(t)$ at the same amplitude of the fundamental mode, since all higher harmonics of this oscillation do not contribute in the resonance case.

If the resonance frequency is not matched, then the limiting case of a small, constant horizontal deflection $y_{\max} = L\theta_{\max} = 2s\delta_{\max}$ is quickly reached which is independent of L, i.e. only about 40cm for s=15cm.

Taking K as a physical pendulum, its eigenfrequency is $\Omega_K = \sqrt{Mgs/J_D}$, with $J_D = J + Ms^2$ as the moment of inertia around D. J_D/Ms is often referred to as the reduced pendulum length L_r, i.e. as the length of the corresponding mathematical pendulum with the same oscillation period as K.
From the literature $J \approx 12\,\mathrm{kgm}^2$ for a body mass of M=70kg is obtained. K generates the deflection δ primarily with the movement of the hip by lifting the legs from the stretched position to a 90°- position and thus becomes a double pendulum himself. The synchronous movement of his lower legs supports the excitation.
Ω_K can be excited particularly easily by K. Ω_K in turn, in order to generate a relevant rope oscillation, must be in resonance with the eigenfrequency $\sqrt{g/L}$ of the rope. This rope length L is the reduced pendulum length L_r of K. With the above J and an assumed s=0.15m one gets $\Omega_K \approx 2.6\,\mathrm{sec}^{-1}$ and $L = L_r \approx 1.5\,\mathrm{m}$. For such short rope lengths L it is easy to start an oscillation.
In order to exite a pendulum oscillation for longer rope lengths, K must move at a lower frequency than his eigenfrequency. This is not easy, but with some skill possible. K only has to carry out the mentioned rectangular oscillation consistently and with the right timing. Children on the swing do that easily. However, K has the problem that, starting from the rest position, he does not know the eigenfrequency of the rope and is typically too fast because he prefers to oscillate at his own eigenfrequency Ω_K . Suppose K has a good sense of time and he matches the eigenfrequency of the rope, then oscillation excitation is possible. But due to the small increase in rope amplitude over time for larger L, a lot of patience is required (see Fig.4) and one inevitably makes mistakes with the many repetitions that make it difficult to build up the oscillation.

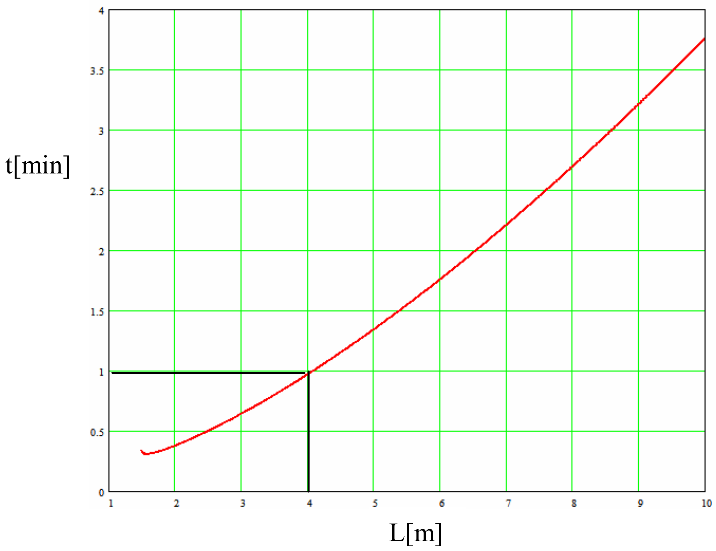

Fig.4: Solving equation (5) for t, one can calculate the required time t as a function of L for a certain y_{max} .
For L=4m with y_{max}=1.5m you already need a minute.

There are also other obstacles. So far, friction and the excitation of other oscillation modes have not been taken into account.

First, the rope has frictional contact with its pivot which dampens the oscillation. When a slightly larger amplitude is reached, the speed of K is no longer negligible and one must take into account both the air friction of the rope and that of K.

If a damping term $\gamma\dot{\theta}$ on the left hand side of the equation of motion (2) is added, then in the case of resonance a maximum amplitude (see y_{max} in Fig.1b) is obtained

$$y_{\max} = L\sin(\theta_{\max}) \approx \frac{J}{ML}\frac{\delta_{\max}}{2}Q \qquad (6)$$

The quality factor Q is given by $Q = \Omega/\gamma$. $Q/2\pi$ is the number of oscillation cycles until the amplitude has dropped to 1/e. y_{max} is reached approximately after the time $1/\gamma = Q\sqrt{L/g}$.

Q can be easily estimated experimentally. This results in quite large Q values between 50 and 100. The oscillation is therefore only weakly damped. If one assumes a quality factor Q=75, then at L=5m you still get a theoretical y_{max}~1.5m. But y_{max} goes to zero with $1/L$. Thus even small frictional losses prevent oscillations with larger amplitudes.

If K is not careful, he can easily stimulate the rotation around the main axis perpendicular to his mediolateral oscillation axis, which leads to a torsion in the rope and which complicates the correct execution of the rectangular swinging. Due to the small restoring torque, this rotational movement is low-frequency and easy to excite.

Another source of interference are the longitudinal oscillations of the elastic rope. These displacements x are described in the linear case by the equation

$$\ddot{x} + \omega_2^2 x = g + s\left(\ddot{\delta}\sin(\delta) + \dot{\delta}^2\cos(\delta)\right) \qquad (7)$$

with $\omega_2 = \sqrt{k/M}$ (see appendix). Like the movement of θ, this is a harmonic oscillation and like equation (2) it is decoupled from the other coordinate. However, here the exciting force depends on s. If the pivot point D coincides with the climber's center of mass S, then x cannot be excited. The solution for x can be found in the appendix (equation (A.7)). In the nonlinear force in equation (7), in addition to Ω, there is also the double frequency 2Ω, so that there are two resonance frequencies. In the case of frequency $2\Omega = \omega_2$ one gets

$$x(t) \cong s\frac{\delta_{\max}^2\Omega}{16}t \cdot \sin(2\Omega t) \qquad (8)$$

In contrast to $\theta(t)$, $x(t)$ depends on the square δ_{max}^2.

ω_2 for a climbing rope can easily be determined from its static elongation which was measured under UIAA standard conditions (indicated by subscript n). First the standard ω_{2n} is determined with the help of Hooke's law $M_n g = k_n L_n \varepsilon_n$. With the standard values for the mass $M_n = 80kg$, the rope length $L_n = 2.6m$ and $\varepsilon_n = 8.5\%$ (averaged over many different single ropes) one gets $\omega_{2n} = 6.66\,sec^{-1}$. This ω_{2n} obtained from a static measurement is significantly smaller than the "dynamic" angular frequency which is responsible for the strength of the impact force [5]. The generalization of the standard ω_{2n} to different M and L results in:

$$\omega_2^2 = \frac{\omega_{2n}^2 M_n L_n}{ML} \qquad (9)$$

If the frequencies Ω and 2Ω are now compared with the eigenfrequencies ω_1 and ω_2 of the rope as a function of its paid-out length L, the following situation arises, which is shown in the next figure.

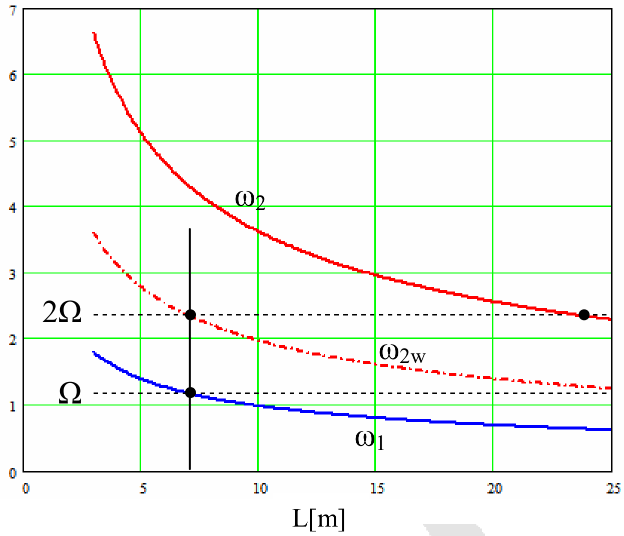

Fig.5: The eigenfrequencies ω_1[sec-1] (blue) and ω_2[sec-1] (red, solid) as a function of L. In order to excite the pendulum oscillation one must select $\Omega=\omega_1$. For L=7m one gets $\Omega\approx1.2$sec-1. In this case of small L the frequency 2Ω that occurs in the force generated by K, is far away from the eigenfrequency ω_2 . For “soft” (half) ropes however, ω_{2w} is the red dashed curve. In this case, both oscillations are excited simultaneously.

For larger L it is easily possible to excite the elastic oscillation instead of the desired pendulum oscillation ω_1. With soft half ropes with significantly smaller ω_{2w}, it is possible that both oscillation modes are excited at the same time.

In rollover swings at fairs and in sports swings, the momentum is generated by the change between a standing and squatting position. The pendulum length is changed periodically, which is called parametric resonance, because one parameter of the oscillation equation (i.e. the pendulum length) changes periodically. This differs from the oscillator discussed here, which is driven by an autonomous force.
Since the center of mass of K periodically moves up and down for $s\neq0$, it is not surprising that this parametric excitation mechanism is also present in the general equation of motion (A.3) for the elastic double pendulum.
The coupling terms of the form $\delta^2\ddot{\theta}$, $\delta\dot{\delta}\dot{\theta}$ and $\delta^2\theta$ (see (A.4), expanded for small δ’s link time-dependent functions with θ or its derivatives. These terms are all zero in the rest position and therefore you cannot leave the rest position with the help of parametric resonance. For small θ the coupling terms are very small and do not matter when exciting the oscillation from rest.
δ either appears in its quadratic form or is multiplied by one of its derivatives. With periodic δ, this leads to excitation functions with twice the frequency (similar to the elastic excitation x), which in turn leads to completely new swing strategies.

Summary

In summary, it can be stated that in principal a freely suspended climber can excite a pendulum oscillation from the rest position, but because of the very low excitation frequency of a longer rope, this only works relatively well with short rope lengths *L* up to approx. 2.5 meters.
The correct way to swing is to take a lying, horizontal position when moving forward at the lowest point, be in an upright sitting position at the reversal point and to keep the upright position at the lowest point when swinging back. Other swinging techniques to excite the pendulum oscillation are not possible, because the parametric excitation mechanism only comes into play for large amplitudes.
In general, the low resonance frequency, which is determined by the pendulum length *L*, must be strictly maintained, which requires a good feeling for the timing of the oscillation. And even if one is able to do this, friction losses due to the suspension of the pendulum and air friction prevent larger deflections. In addition, the pendulum amplitude increases only very slowly with larger *L*, so that a lot of patience is required.
After a small fall under an overhang there is a good chance of coming back to the wall if one tries to amplify the pendulum motion as quickly as possible using the swing strategy discussed above and not waiting for the oscillation to attenuate.
Already at a rope length of 5 meters, the author thinks that there is no chance to reach even a meter of deflection from the rest. Ambitious readers are encouraged to find out the rope lengths up to which they are able to put themselves into a relevant pendulum swing from their rest position (e.g. in the climbing gym). The author is looking forward to your feedback.
In order not to get into the situation of hanging freely and motionless during an abseil maneuver, as soon as it becomes overhanging it is necessary to swing by pushing the legs off the rock, to clip quickdraws in hopefully existing bolts and to stay in a pendulum swinging by continuously pushing away from the wall. If it does happen, it’s not worth starting senseless attempts again and again and thus wasting power, it only helps to prusik (or to be rescued).

Acknowledgement

The author thanks Ira Leuthäusser for her critical objections and for many helpful discussions.

Appendix

The Lagrangian function $\tilde{L}$ [6] for the elastic double pendulum (see Fig.1b) is given by

$\tilde{L}$ = kinetic energy - potential energy =

$$\frac{M}{2}\left(r^2\dot{\theta}^2+\dot{r}^2\right)+\frac{1}{2}J_D\left(\dot{\theta}+\dot{\delta}\right)^2+Ms\left[r\dot{\theta}\left(\dot{\theta}+\dot{\delta}\right)cos(\delta)-\dot{r}\left(\dot{\theta}+\dot{\delta}\right)sin(\delta)+g\,cos(\theta+\delta)\right]+ \\ Mgr\,cos(\theta)-\frac{1}{2}k(r-L)^2 \tag{A.1}$$

Here are:

r=L+x the total length of the rope, composed of the unstretched length L and the elongation x

M the mass of the climber K

J_D=J+Ms^2 the moment of inertia around the pivot point D

J the moment of inertia around the center of mass S

θ the angle formed by the rope with the vertical

δ the angle between K and rope

s the distance between pivot point D and the center of mass S

k the spring constant of the rope.

The equation of motion for θ is obtained from the Lagrangian equations [6] for this coordinate

$$(Mr^2+J_D+2Msr\,cos(\delta))\ddot{\theta}+2Mr\dot{r}\dot{\theta}+2Ms\dot{r}\dot{\theta}\,cos(\delta)-2Msr\dot{\theta}\,sin(\delta)\dot{\delta}-Ms\ddot{r}\,sin(\delta) \\ -Ms\dot{r}\,cos(\delta)\dot{\delta}+Msg\,sin(\theta+\delta)+Mrg\,sin(\theta)=-J_D\ddot{\delta}-Msr\,cos(\delta)\ddot{\delta}+Msr\,sin(\delta)\dot{\delta}^2 \tag{A.2}$$

There is a linear coupling to the x coordinate through the term $Ms\dot{x}\,cos(\delta)\dot{\delta}$.

For $\dot{r}=0$ and small angles θ it follows from (A.2)

$$(Mr^2+J_D+2MsL\,cos(\delta))\ddot{\theta}-2MsL\dot{\theta}\,sin(\delta)\dot{\delta}+Msg\theta\,cos(\delta)+MLg\theta= \\ -Msg\,sin(\delta)-J_D\ddot{\delta}-MsL\,cos(\delta)\ddot{\delta}+Msr\,sin(\delta)\dot{\delta}^2 \tag{A.3}$$

The equations of motion (A.2) and (A.3) are restricted by the (holonomic) constraint δ=f(t) and are equations for the variable θ alone with time-dependent coefficients given by δ. The terms of θ and its derivatives, which are linked to a time-dependent function and therefore are responsible for the parametric resonance, are

$$2MsL\,cos(\delta)\ddot{\theta},\; 2MsL\dot{\theta}\,sin(\delta)\dot{\delta} \text{ and } Msg\theta\,cos(\delta) \tag{A.4}$$

The Lagrangian equation for *x* is given by

$$M\ddot{x} - Ms\left(\ddot{\theta} + \ddot{\delta}\right)sin(\delta) - Ms\left(\dot{\theta} + \dot{\delta}\right)^2 cos(\delta)\dot{\delta} - Mr\dot{\theta}^2 - Mg\,cos(\theta) + kx = 0 \qquad (A.5)$$

For $\theta << \delta$ and neglecting all quadratic and higher terms in $\dot{\theta}$ and θ, one obtains

$$\ddot{x} + \frac{k}{M}x = g + s\left(\ddot{\delta}\,sin(\delta) + \dot{\delta}^2\,cos(\delta)\right) \qquad (A.6)$$

The solution for *g*=0 is

$$x(t) = \left(\frac{\delta_{max}}{2}\right)^2 s\cdot\Omega^2\left[\frac{cos(2\Omega t)}{(2\Omega)^2 - \omega_2^{\,2}} - \frac{cos(\Omega t)}{\Omega^2 - \omega_2^{\,2}} + \frac{3\Omega^2\,cos(\omega_2 t)}{\left((2\Omega)^2 - \omega_2^{\,2}\right)\left(\Omega^2 - \omega_2^{\,2}\right)}\right] \qquad (A.7)$$

with $\omega_2 = \sqrt{k/M}$. The pendulum motion $\delta(t) = \frac{\delta_{max}}{2}(1 - \cos(\Omega t))$ generates a non-linear force in (A.6), which also contains a term of twice the frequency 2Ω. This leads to two resonances, either $2\Omega = \omega_2$ or $\Omega = \omega_2$.

References

[1] W.B.Case and M.A.Swanson, “The pumping of a swing from a seated position” Am.J.Phys. 58 (5) (1990)
[2] W.B.Case, “The pumping of a swing from the standing postion”, Am.J.Phys. 64 (3) (1996)
[3] S.Wirkus, R.Rand and A.Ruina, “How to pump a swing”, The College Math. J. 29,4 (1998)
[4] A.Post, G. de Groot, A. Daffertshofer and P.Beek, ”Pumping a playground swing” Motor Control 11 (2007)
[5] U.Leuthäusser, “The physics of a climbing rope under a heavy dynamic load”, Proc. Of the Institution of Mechanical Engineers Part P 231 (2) (2017)
[6] D. Morin, *Introduction to Classical Mechanics*, Cambridge University Press (2007)