The benefits of the utility function approach: a plausible explanation for small risky parts in a portfolio

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(20.07.2011)

We present an approach using a utility function for a large class of risk-averse investors who want to determine what fraction of their portfolio they should expose to risk. It is shown that even for long investment horizons the risky asset portion is relatively small, varying only between 0.2 and 0.3 without depending much on the investor's risk aversion.

The utility function used consists of a linear component in the rate of return and a strongly decreasing component for negative rate of returns. The expectation values of the utility function are calculated with the probability distribution of a Geometric Brownian Motion, the common model of a stock market. Common quadratic approximations, i.e. an analysis by the mean value and the variance only, are not able to reproduce the results of this article because of the long-tail properties of the lognormally distributed rates of return.

Furthermore, the same utility function is used for the selection of more complicated investments like discount or bonus certificates. Depending on the investor's risk aversion, his "optimal" parameters like cap value, bonus level, etc. are calculated.

1. Concept

The utility function concept[1] is used to evaluate the outcome of a set of actions (different investment strategies). Each action a_j from n possible actions results in a certain return $R(a_j) = \frac{w(a_j)}{w_0} - 1$ and is valued by a utility function U(R). w_0 is the initial wealth, and $w(a_i)$ the wealth obtained, if the action a_j is chosen. R is a random variable with a probability distribution $p(R|a_j)$, conditionally dependent on a_j. Taking into account all possible R's, the conditional expectation $E[U(R)|a_j]$ for a given a_j is given by

$$E[U(R) \mid a_j] = \int U(R) \cdot p(R \mid a_j) dR \tag{1}$$

The best strategy a_{opt} is given by

$$a_{opt} = \arg\max_{1 \le j \le n} \left(E[U(R) \mid a_j] \right) \tag{2}$$

The action variable a_j which here is assumed as discrete can also be continuous. U is defined only up to linear transformations, because $U(R) \to cU(R) + b$ (c>0) leads to the same preference. This is reminiscent of describing physical systems in static equilibrium where U is the potential energy.

U can be either a function of R alone or depends also on the initial wealth w_0. For $U(w) = -\exp(-Aw)$, for example, one obtains $U(w) - U(w_0) = -\exp(-Aw) + \exp(-Aw_0) = \exp(-Aw_0)(1 - \exp(-ARw_0)) \propto 1 - \exp(-A'R)$, i.e. the parameter A'=$Aw_0$, the aversion against risk, increases with increasing initial wealth w_0. In contrast one obtains for a logarithmic $U(w)-U(w_0)$ = $\ln(w) - \ln(w_0) = \ln(\frac{w}{w_0}) = \ln(1+R)$ which depends only on the return R, independent of the absolute level of wealth.

The so-called certainty equivalent C which solves the equation

$$U(C) = E[U(R) \mid a] \tag{3}$$

[1] for a more complete overview, see for example:
Levy, H. and Sarnat, M., *Portfolio and Investment Selection*, Prentice Hall, 1984
Berger, J.O., *Statistical Decision Theory and Bayesian Analysis*, Springer, 1985

is an interesting concept and can be used to find the personal attitude to risk. For risk-averse investors, U(R) is concave and therefore C is always smaller than E(R|a). C is the riskless rate of return which is equivalent to the risky E[R]. The difference

$$\Pi = E[R|a] - C \tag{4}$$

is called the risk premium which the investor is willing to pay to get rid of the risk.

In order to get an approximate expression for Π, the left hand side of (3) is expanded for small Π:

$$U(C) = U(E[R \mid a] - \Pi) \cong U(E[R \mid a]) - \Pi U'(E[R \mid a]) + \frac{1}{2}\Pi^2 U''(E[R \mid a])$$

and, by expanding U around E[R], the right hand side of (3) can be approximately written as

$$E[U(R) \mid a] \cong U(E[R \mid a]) + \frac{\sigma_R^2}{2} U''(E[R \mid a]) \,.$$

σ_R^2 is the variance of the distribution p(R|a). Π can now be calculated and is given by

$$\Pi \cong \frac{U'(E[R])}{U''(E[R])}\left(1 - \sqrt{1 + \sigma_R^2 \left(\frac{U''(E[R])}{U'(E[R])}\right)^2}\right) \tag{5}$$

Expansion for small σ_R leads to the approximation $\Pi_{Pratt} = -\frac{\sigma_R^2}{2}\frac{U''(E[R])}{U'(E[R])}$. Economists spend a name for this expression and call it Arrow-Pratt risk aversion function. It is important to note that these approximations must be applied carefully. Whereas (5) is exact for quadratic utility functions, Pratt's approximation gives only poor results for large σ's. For a lognormal distribution the expectation of the "Bernoulli" utility ln(1+R) is exactly known, i.e. $E[\ln(1+R)] = \int \ln(1+R) \cdot \mathrm{Ln}\,(1+R, \mu^*, \sigma^*) dR = \mu^*$, with the possibility to test the approximations.

In the following we discuss the utility with aversion parameter B

$$U(R) = R + \frac{1}{B}(1 - \exp(-BR)) \tag{6}$$

shown in Figure 1:

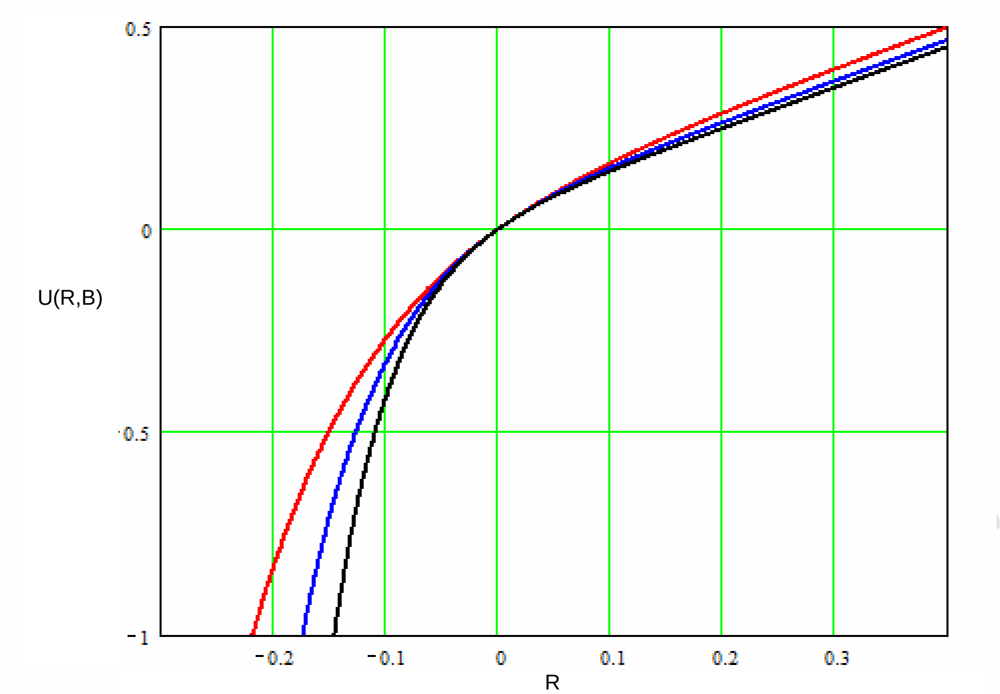

Fig.1: A larger B leads to a stronger drop in U for negative R (red: B=10, blue: B=15, black: B=20). For positive R, the influence of parameter B is not very significant.

We believe that the utility U for the discussed problem is only weakly dependent on the absolute w and mainly determined by R. There is a linear preference for positive returns and for larger B >> 0 a strong aversion for negative returns. The linear increase in R describes the attitude of an investor better than flattening U's like the logarithmic one or the often used quadratic $U(R) = R - \frac{1}{2}cR^2$. The latter even decreases for larger R, unreasonably confining the investor's preference for large returns. Using such a U, one already anticipates the statistical properties of R. Because a higher R is connected with a larger spread in its distribution function expressing greater risk, one obtains reasonable results for the expectation value of the utility. But then the modelling could be started directly with E[U] without an underlying U(R).

The calibration of U(R) for a specific investor is possible in several ways. One possibility takes the opposite cases R =10% and R=-10% and estimates the investor's advantage resp. disadvantage of these R's.

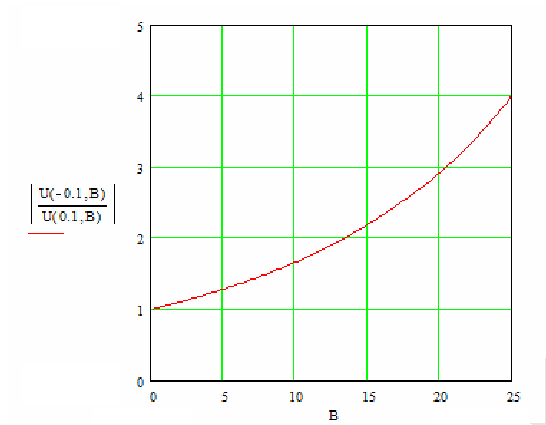

Fig.2: Absolute ratio of the utility for R=-10% and R=10% as a function of B

If, in an absolute sense, the loss is as painful as the pleasure of the gain, we have a risk neutral investor (B=0). However, usually investors are risk-averse and typical values of B are about 13 (see Figure 2): the damage is about twice as high as the gain.

2. Determination of the fraction of wealth exposed to risk

In the context of portfolio selection an investor has to decide which part of his wealth should be exposed to risk in order to get an increased return.

We assume that the price $p_1(t)$ of a stock market follows a Geometric Brownian Motion with the stochastic equation

$$dp_1(t) = p_1(t) \cdot [\mu_r dt + \sigma_r dW(t)] \qquad (7)$$

Applying the Ito calculus, $y=\ln(p_1(t)/p_1(0)= \ln(1+r(t))$ has a Normal probability distribution

$$p_y(y,t) = \frac{1}{\sqrt{2\pi}\sigma_r \sqrt{t}} \exp\left(- \frac{\left(y - \left(\mu_r - \frac{1}{2}\sigma_r^2\right)t\right)^2}{2\sigma_r t} \right)$$

Thus r is lognormally distributed:

$$r \sim p_r(r,t) = \mathrm{Ln}\left(r+1, (\mu_r - \frac{1}{2}\sigma_r^2)t, \sigma_r\sqrt{t} \right) \qquad (8)$$

The relations

$$\mu_r = \frac{1}{t_y}\ln(1+E[r(t_y)]) \qquad \text{and} \qquad \sigma_r^2 = \frac{1}{t_y}\ln\left(1+\frac{V[r(t_y)]}{(1+E[r(t_y)])^2}\right)$$

allow to calculate μ_r and σ_r from the input parameters $E[r(t_y)]$ as the mean annual rate of return and $V[r(t_y)]$ as the annual variance.

The price of the riskless asset has the differential equation

$$dp_0(t) = p_0(t)\cdot\mu_0 dt \tag{9}$$

Taking into account the inflation rate μ_i, we use the real, inflation adjusted rates $\mu_r - \mu_i$ and $\mu_0 - \mu_i$ instead of the nominal rates μ_r and μ_0 without introducing new variables.

Assuming a portfolio of value w formed by the two assets with N_1 units of risky asset and N_0 units of the riskless asset (e.g. saving account), the change of that portfolio is given by

$$dw = N_0 dp_0 + N_1 dp_1 \tag{10}$$

for constant N_0 and N_1. This equation can be written as

$$\frac{dw}{w} = (1-x(t))\cdot p_0(t)\mu_0 dt + x(t)\cdot p_1(t)[\mu_r dt + \sigma_r dW(t)] \tag{11a}$$

with the time-dependent fraction of the risky asset

$$x(t) = \frac{N_1 p_1(t)}{N_1 p_1(t) + N_0 p_0(t)} \tag{12}$$

The exact solution of equation (10) for fixed N_0 and N_1 is obtained by writing it in the form

$$dw = \frac{(1-x(0))w(0)}{p_0(0)}dp_0 + \frac{x(0)w(0)}{p_1(0)}dp_1 \tag{11b}$$

which can be integrated:

$$\frac{w(t)-w(0)}{w(0)} = (1-x(0))(\exp(\mu_0 t)-1) + x(0)\left(\frac{p_1(t)}{p_1(0)}-1\right) \tag{12a}$$

or

$$R(t) = (1-x(0))(\exp(\mu_0 t)-1) + x(0)r(t) \tag{12b}$$

We know that R(t) is lognormally distributed, therefore

$$\tilde{p}_R(R,t) = \frac{1}{x(0)}\,\mathrm{Ln}\left(1+\frac{R(t)-(1-x(0))(\exp(\mu_0 t)-1)}{x(0)}, (\mu_r - \frac{1}{2}\sigma_r^2)t, \sigma_r\sqrt{t}\right) \tag{13}$$

It is also possible to solve (11a) applying an additional steady control process for N_0 and N_1 in order to get a constant x. After price changes between time t and t'=t+dt one obtains a new w(t') and a changed intermediate $\tilde{x}$. In order to undo this change and to get x = const, N_1(t') and N_0(t') have to be modified in the following way

$$\begin{aligned} N_0(t') &= \frac{w(t')}{p_0(t')}(1-x(0)) \\ N_1(t') &= \frac{w(t')}{p_1(t')}x(0) \end{aligned} \tag{14}$$

Note that the new N_1(t') and N_0(t') do not change w(t') (otherwise a third component like cash would appear).

Under these conditions the return of the portfolio R=w(t)/w(0)-1 (equations 11 with x=const) is lognormally distributed like r and given by

$$p_R(R,t\,|\,x) = \mathrm{Ln}\left(R+1, (\mu_0(1-x)+x\mu_r - \frac{1}{2}\sigma_r^2 x^2)t, x\sigma_r\sqrt{t}\right) \tag{15}$$

The expectation, variance and mode (the point where the probability density is maximal) are given by

$$E[R \mid x] = \exp((\mu_0(1-x) + x\mu_r)t) - 1$$

$$V[R \mid x] = \exp(2(\mu_0(1-x) + x\mu_r)t)(\exp(x^2\sigma_r^2) - 1) \qquad (16)$$

$$R_{mod} = \exp\left((\mu_0(1-x) + x\mu_r - \frac{3}{2}\sigma_r^2 x^2)t\right) - 1$$

The most probable return has a maximum at $\hat{x} = \frac{1}{3}\frac{\mu_r - \mu_0}{\sigma_r^2}$ which is surprisingly independent of t. When $R_{mod}(x)$ is plotted against the loss probability $P_{loss}(x) = \int_{-1}^{0} p_R(R,t \mid x)dR$, one finds for small x a strongly increasing $R_{mod}(x)$ with only a little change in $P_{loss}(x)$. However, for larger x approaching $\hat{x}$, P_{loss} grows much more than R_{mod} leading mainly to higher risk without higher return. For the determination of the optimal x, R_{mod} is therefore not as appropriate as the utility approach.

In Figure 3 several numerically calculated expectation values of U(R) (equation 6)

$$E[U(R) \mid x] = \int U(R) \cdot p_R(R,t \mid x)dR$$

are shown as a function of the portion x of the risky asset.

The following rates of return (per day) have been used: $\mu_r = 1.501 \cdot 10^{-4}$, $\mu_0 = 2.787 \cdot 10^{-5}$ corresponding to annual rates of return (inflation adjusted) of 5.3% and 0.98%. The standard deviation is $\sigma_r = 9.86 \cdot 10^{-3}$ leading to an annual volatility of 20%.

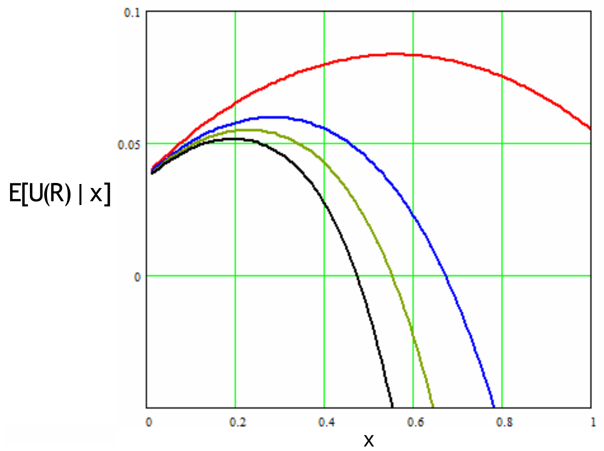

Fig.3: The expected utility has a maximum with respect to x which depends on the aversion parameter B (red: B=5, blue: B=10, green: B=12.5, black: B=15). t=2 years.

The red curve in the following Figure 4 shows the expected return of a 2 year investment as a function of the parameter B. An investor with a higher B is more risk-averse. Thus the red curve varies between a full investment (x=1) with an average return of the risky asset alone and an investment mostly in the risk-free asset with μ_0. The blue curve is the certainty equivalent. For example for B=10, we have $E[R\,|\,x] = \exp\left((\mu_0(1-x_{opt}) + x_{opt}\mu_r)t\right) - 1$ = 4.4% and CE=3.2%. Thus for this investor a riskless rate of return of 3.2% is equivalent to the expected E[R|x] = 4.4% which is only obtained on average and he would pay 1.2% in order to avoid the uncertainty.

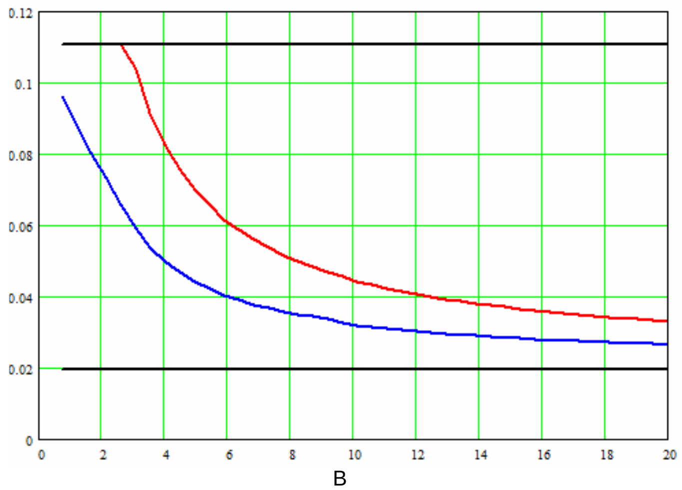

Fig.4: red: $E[R\,|\,x]$, blue: certainty equivalent C, upper black: full investment E[R|1], lower black: E[R|0] as functions of B.

In Figure 5a,b the optimal x as a function of B are shown.

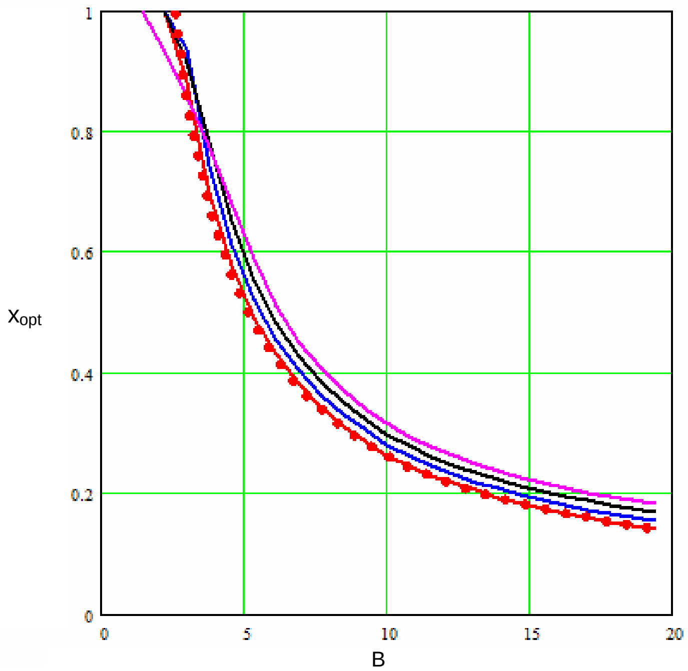

Fig.5a: Optimal fraction of the risky asset x for different investment times (red: t=1y, blue: t=2y, black t=3y and magenta t=4y) and as dots the approximation (17) as a function of B.

An approximation for x_{opt} shown in the Figure 5a as red dots which is excellent for smaller returns resp. shorter investment horizons is given by

$$x_{opt} \cong \frac{1+e^{B\mu_0 t}}{B} \cdot \frac{\mu_r - \mu_0}{\sigma_r^2} \qquad (17)^2$$

For values of B smaller than 4 one obtains a full investment in the risky asset. However, in the relevant range of B (magnified in the following Figure 5b) there is neither a strong dependence on B nor a strong dependence on the investment time. Assuming a medium risk aversion which is valid for many investors, we have the important result that x varies between 0.2 and 0.3 with only a slight dependence on time. Although the expected return of the stock market is growing with time, the variance increases also, compensating the rising expectation. This differs from the popular and widespread opinion that high x are justified for long investment horizons. However, as can be seen from the approximate x_{opt} (equation 17), an investor who expects a bull market in the next future with a higher than

[2] The expression $(\mu_r - \mu_0)/\sigma_r^2$ can be obtained immediately when an expected utility $E[U] = \mu_0 t + x(\mu - \mu_0)t - \frac{B}{4}x^2\sigma_r^2 t$ is used that consists only of the first two moments.

average return μ_r of the stock market will choose a higher optimal x, because x rises linearly with increasing μ_r.

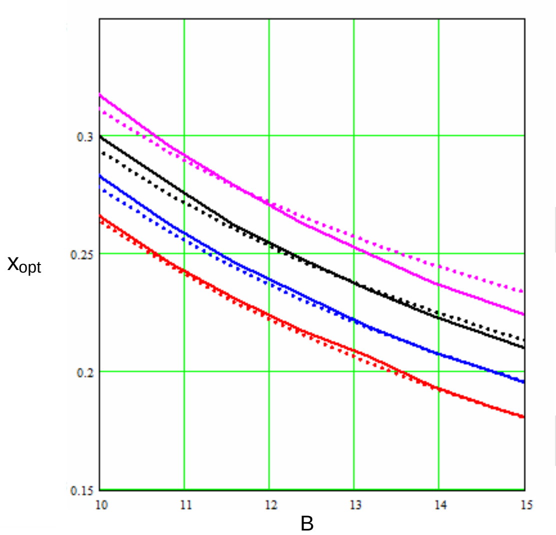

Fig.5b: Optimal x for several investment times (red: t=1y, blue: t=2y, black: t=3y, magenta: t=4y). The dotted curves are the corresponding approximations (17).

3. Optimal choice between several investments

So far we have determined the optimal fraction of one risky asset depending on the type of the investor.

In principle, the next step should include several assets with the following optimization problem $\vec{x}_{opt} = \arg(\max(\int U(\vec{R} \mid \vec{x}) p(\vec{R}) d\vec{R}))$, where the components of $\vec{x}$ sum up to one. Conceptually simple, but elaborate, we instead look for the best decision for an investor who wants to buy a specific security for a certain amount of money. One could argue that this money comes from the risky part and therefore the investor is risk neutral for that part. In this case U(R) is proportional to R, and therefore one should take the asset with the largest expectation value. But in general, the investor wants to choose between a wider range of B's, keeping in mind that for this problem the aversion parameter B has to be adjusted and is usually smaller than in our first problem.

First, we investigate **discount certificates**. The investor who wants to buy such a certificate has to choose a cap value according to his risk behaviour. A wide range of cap values is available, varying from low caps (simulating a saving account) up to large caps where the discount certificate is essentially equivalent to the underlying asset. Thus, the possibility to buy the underlying asset is included.

In Figure 6 a typical probability distribution $P_D(R_D)$ of a discount certificate is shown. It is characterized by a delta function at the cap and a truncated Lognormal distribution left of it[3].

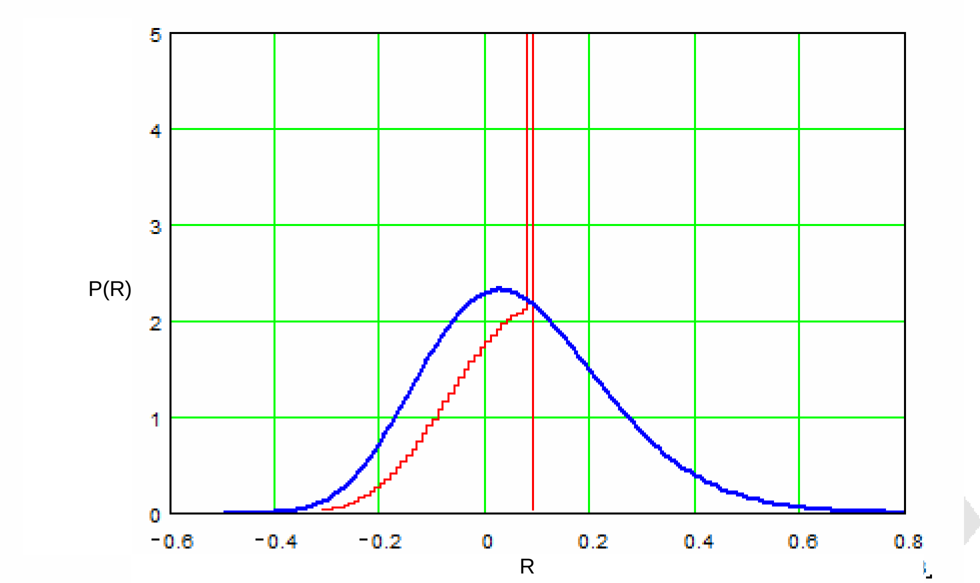

Fig.6: Probability density of a discount certificate (red) and its underlying stock (blue) as a function of return R.

In the next Figure 7 the expected utility $E[R_D]$ using P_D is numerically calculated. Depending on the investor's risk aversion parameter B, the maxima of $E[R_D]$ specify his optimal cap of the discount certificate.

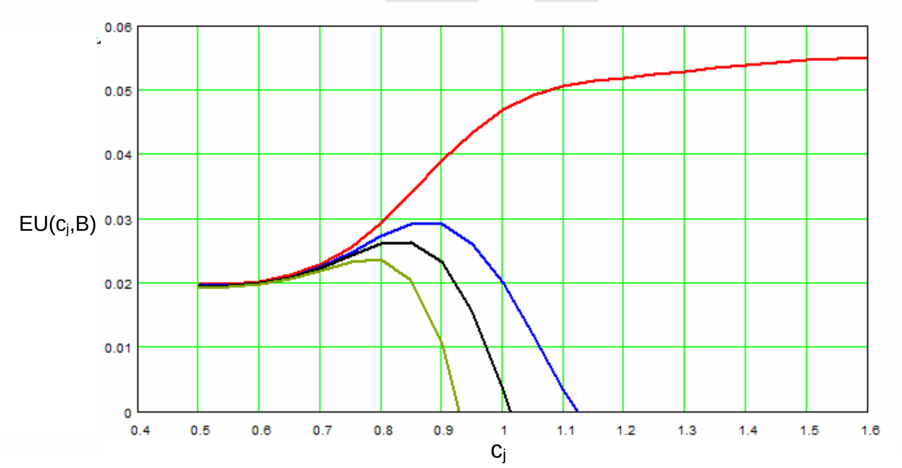

Fig.7: An investor with B=5 (red) buys the underlying asset. An investor who is more risk-averse with B=10 (blue) prefers a discount certificate with a cap value of about 0.9. An even larger B (B=12 black; B=15 green) shifts the maximum of the expected utility to smaller values of the cap c_j (c_j is normalized by the present value J of the underlying asset).

[3] U. Leuthäusser: Theoretische Beschreibung von Discountzertifikaten, October 2010; http://www.sigmadewe.com/fileadmin/user_upload/pdf-Dateien/Theorie_Discountzertifikate.pdf

Second, we shortly discuss the decision to buy a stock or a stock index directly or their associated **bonus certificates**. A bonus certificate is characterized by a bonus level b and an absorbing barrier a[4]. In Figure 8 the return probability distribution of a bonus certificate with its typical split distribution and the probability distribution of the corresponding underlying asset are shown. The parameters a=-0.45 and b=0.125 are a risk-averse choice which excludes completely moderate to high losses (unfortunately a small probability of a very high loss remains) and therefore only a low average return can be expected.

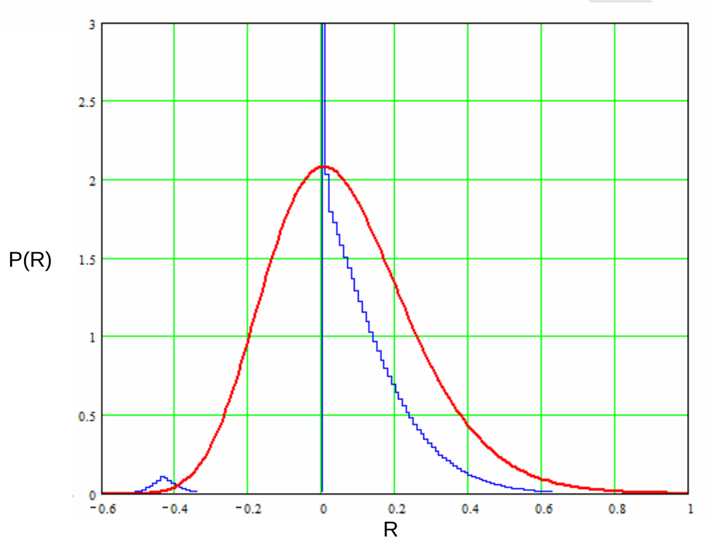

Fig.8: Probability density of a bonus certificate (blue) and its underlying stock (red).

In Figure 9 the utility of the underlying asset U and the utility of the corresponding bonus certificate U_B (a=-0.45, b=0.125) is shown. For that choice of parameters the bonus certificate has an advantage over the underlying in a large range of B.

[4] U. Leuthäusser: Theoretische Beschreibung von Bonuszertifikaten, February 2011; http://www.sigmadewe.com/fileadmin/user_upload/pdf-Dateien/Theorie_Bonuszertifikate.pdf

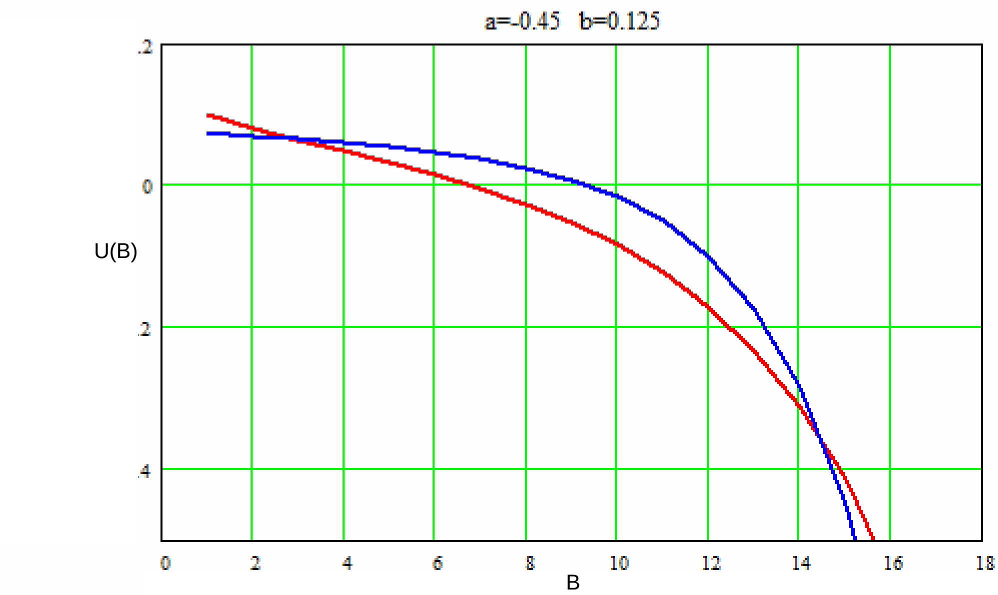

Fig.9: The expected utility of a bonus certificate (blue) and the corresponding expected utility of the underlying asset as a function of the aversion parameter B.

In this paper we found that our utility function is very helpful for the composition of a portfolio. For typical investors it leads to rather small risk exposed parts of the portfolio which increase only slowly with the investment horizon. Furthermore, the last two examples show that the presented utility easily enables to assess even more complicated (option-based) investments and to find out their appropriate parameters.