Physics of climbing ropes: impact forces, fall factors and rope drag

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Introduction

In a previous work [1] the behaviour of a climbing rope in the UIAA heavy fall case is presented.

One result of this paper was that viscous friction can be approximately neglected until the force maximum and irrespective of nonlinear forces a harmonic oscillator model is a good approximation.

On this basis it is shown in the following how a climbing rope behaves in the case of one or more protection points (usually bolts with quickdraws) between leader and belayer taking into account the so-called dry friction between rope and the protection points.

It turns out that dry friction leads to the same form of equations and thus the same expressions for dynamic elongation and impact force as in the case without dry friction, if the elastic modulus is redefined properly.

Furthermore the rope drag is calculated depending on the number of protection points and the angle deviations of the rope at these points.

Other work on this subject can be found in [2] with a numerical simulation approach. [3] presents experimental data of drop tests.

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[1] U. Leuthäusser, The physics of a climbing rope, *www.sigmadewe.com/fileadmin/user_upload/pdf-Dateien/Physics_of_climbing_ropes.pdf*
[2] M. Pavier, Experimental and theoretical simulations of climbing falls, Sports Engineering (1998) **1**, 79-91
[3] J. Marc Beverly, Stephen W. Attaway, Measurement of Dynamic Rope System Stiffness in a Sequential Failure for Lead Climbing Falls, *http://www.mra.org/images/stories/members/Beverly_Sequential_Falls2.pdf*

1. Friction between rope and protection points (harmonic oscillator model)

We consider the situation depicted in the figure below. The leading climber takes a fall of a distance $2l_n$ above the last protection P_{n-1}. At the end of the fall when the rope begins to stretch he has a velocity $v_0 = \sqrt{4gl_n}$ and his location is called x_n. The rope responds to the fall with elongations x_i at P_i (i=1, ..., n-1). The rope segments and their spring constants between P_{i-1} and P_i are denoted as l_i and k_i. Together with the friction constant μ the angle α_i at P_i determines the friction force at this point.

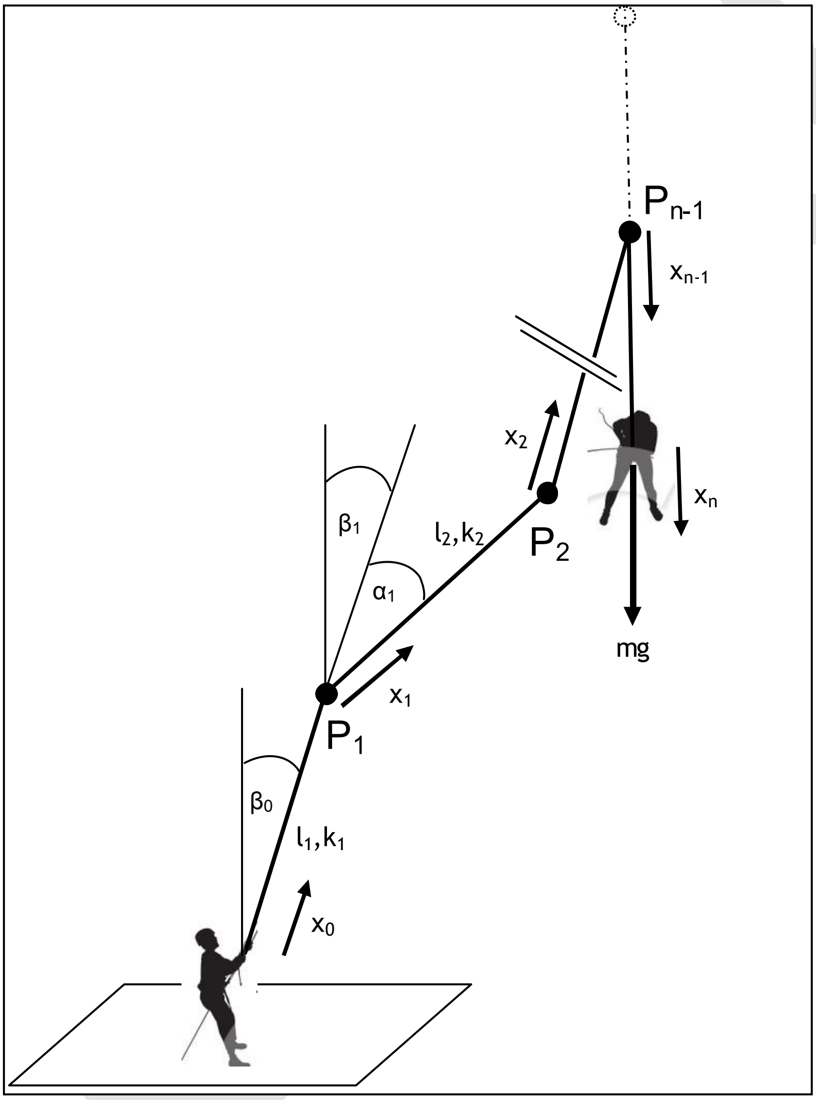

Fig. 1

To keep things relatively simple, we use a harmonic oscillator (HO) model with the Lagrange function

$$L = \frac{m}{2}\dot{x}_n^2 - \left[\frac{1}{2}k_1(x_1 - x_0)^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \ldots + \frac{1}{2}k_n(x_n - x_{n-1})^2 - mgx_n\right] \tag{1}$$

The coordinate x_0 can be controlled by the belayer. For a static belayer x_0 is zero.

The Lagrange equations for non conservative systems are given by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = F_i \tag{2}$$

with the dissipative forces

$$F_i = k_i(x_{i+1} - x_i)(1 - e^{-\mu\alpha_i}) \tag{3}$$

using the equation of Euler-Eytelwein (see Figure 2). Their dependency on the direction of motion is omitted here, thus they are only valid for short times including, however, the times of maximum elongation and acceleration. A more detailed discussion of dry friction can be found in appendix A.

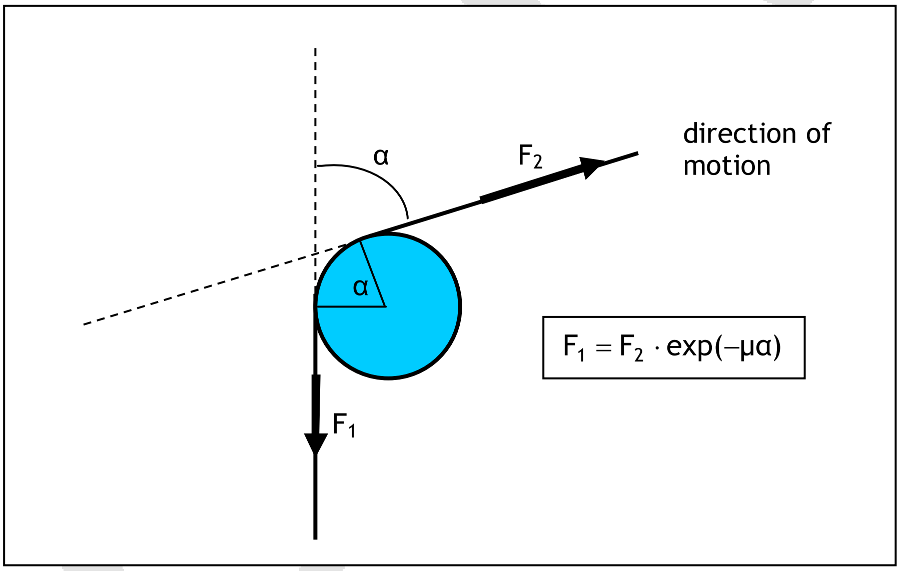

Fig. 2. When a force F_2 pulls a rope over a curved surface with friction μ, the force F_1 on the opposite side is reduced and given by $F_1 = F_2 \exp(-\mu\alpha)$. This force depends on the contact angle α between rope and surface (formula of Euler-Eytelwein), but not on the curvature of the surface.

Equations (2) and (3) immediately lead to

$$
\begin{aligned}
&k_2(x_2 - x_1)e^{-\mu\alpha_1} - k_1(x_1 - x_0) = 0 \\
&k_3(x_3 - x_2)e^{-\mu\alpha_2} - k_2(x_2 - x_1) = 0 \\
&\vdots \\
&k_i(x_i - x_{i-1})e^{-\mu\alpha_{i-1}} - k_{i-1}(x_{i-1} - x_{i-2}) = 0 \\
&\vdots \\
&k_n(x_n - x_{n-1})e^{-\mu\alpha_{n-1}} - k_{n-1}(x_{n-1} - x_{n-2}) = 0 \\
&m\ddot{x}_n + k_n(x_n - x_{n-1}) = mg
\end{aligned}
\tag{4}
$$

This system of equations can also be obtained without the formalism of Lagrange. The last equation is a HO equation for the mass m. No other masses are involved, thus the equations represent the balance of forces at the protection points considering friction.

It is possible to get an equation for the elongation x_n of the form

$$
m\ddot{x}_n + k_{eff}(x_n - x_0) = mg \tag{5}
$$

from the system (4). In order to eliminate all the intermediate x_i, we solve the first equation for x_1, substitute it in the second equation which leads to

$$
k_3(x_3 - x_2) = \frac{x_3 - x_0}{\dfrac{1}{k_3} + \dfrac{1}{\rho_2 k_2} + \dfrac{1}{\rho_1 \rho_2 k_1}}
$$

with the abbreviation $\rho_i = \exp(\mu\alpha_i)$. Continuing this procedure one obtains

$$
k_n(x_n - x_{n-1}) = \frac{x_n - x_0}{\dfrac{1}{k_n} + \dfrac{1}{\rho_{n-1} k_{n-1}} + \ldots + \dfrac{1}{\rho_1 \rho_2 \ldots \rho_{n-1} k_1}}
$$

Comparison with equation (5) yields the effective spring constant

$$
\frac{1}{k_{eff}} = \frac{1}{k_n} + \frac{1}{\rho_{n-1} k_{n-1}} + \ldots + \frac{1}{\rho_1 \rho_2 \ldots \rho_{n-1} k_1} \tag{6}
$$

The k_i depend on the lengths l_i:

$$
k_i = \frac{E \cdot q}{l_i} \tag{7}
$$

where E is the elastic modulus and q the cross section of the rope. Substituting the k_i into equation (5), it reduces to

$$
k_{eff} = \frac{Eq}{l_n + \rho_{n-1}^{-1} l_{n-1} + \ldots + \rho_1^{-1} \rho_2^{-1} \ldots \rho_{n-1}^{-1} l_1} = \frac{Eq}{l_{eff}} \tag{8}
$$

with an effective rope length

$$l_{eff} = l_n + \rho_{n-1}^{-1} l_{n-1} + \ldots + \rho_1^{-1}\rho_2^{-1}\ldots\rho_{n-1}^{-1} l_1 < l$$

that is always smaller than l. The formula for k_{eff} is very intuitive and can almost be guessed: if there is no friction (all $\rho_i = 1$), k_{eff} is given by $\frac{1}{k_{eff}} = \sum_{i=1}^{n} \frac{1}{k_i}$, the well known formula for springs in series. In this case we find the lowest possible value for $k_{eff} = Eq/l$, the spring constant of a rope with length l. In the opposite case $\rho_{n-1} >> 1$ of large friction (i.e. the rope is pinned at the last protection point) only the last part of the rope acts as an oscillator and k_{eff} obtains its maximum possible value $k_{eff} = Eq/l_n$ without any energy dissipation. In between, the spring constant k_{n-1} is replaced by a larger effective spring constant $\rho_{n-1} \cdot k_{n-1}$, k_{n-2} replaced by $\rho_{n-1}\rho_{n-2} \cdot k_{n-2}$, and so on. Thus dry friction always leads to a higher spring constant. Taking UIAA norm fall conditions, the spring constant is only slightly increased, because of the very small $l_1 = 0.3$ m compared to the total $l = 2.6$ m

$$k_{eff} = \frac{e^{\mu\alpha} \cdot k_1 k_2}{e^{\mu\alpha} \cdot k_1 + k_2} \approx \frac{Eq}{l}\left(1 + \left(1 - e^{-\mu\alpha}\right)\frac{l_1}{l}\right) = \frac{Eq}{l}(1 + 0.06)$$

with $\mu = 0.25$ and $\alpha = \pi$. Thus the error in neglecting the friction between rope and carabiner is about 6%.

In the limiting case of infinite friction $\mu \to \infty$ one gets $x_0 = x_1 = \ldots\ x_{n-1}$, which are not necessarily zero, because we have neglected the mass of the rope. Taking into account the rope mass the motion of x_1 is prevented, i.e. $x_1 = \ldots\ x_{n-1} = 0$. Because the rope is flexible, a rope feed from the belayer at x_0 leads only to a slack rope but not to a motion of x_1.

Taking the expression for the impact force on a rope in the HO model [1]

$$F^{max} = mg + m \cdot \max(|\ddot{x}_n|) = mg + \sqrt{2mghk_{eff} + m^2 g^2} = mg + \sqrt{2mgh\frac{Eq}{l_{eff}} + m^2 g^2} \qquad (9)$$

an effective fall factor can now be defined as

$$f_{eff} = f\frac{l}{l_{eff}} = \frac{h}{l_{eff}}$$

where $f = h/l$ is the usual fall factor without friction. F^{max} with friction divided by F^{max} without friction ($l_{eff} = l$) scales approximately like $\sqrt{l/l_{eff}} \geq 1$.

The frictional force on the last protection point n-1 is given by

$$F_{n-1} = k_n(x_n - x_{n-1})(1 - e^{-\mu\alpha_{n-1}})$$

For its maximum value $F_{n-1}^{max} = F^{max}(1 - e^{-\mu\alpha_{n-1}})$ and with typical values for μ = 1/4 and $\alpha_{n-1} \approx \pi$, we find $F_{n-1}^{max} = 0.544 \cdot F^{max}$.

The corresponding fall factor is given by $f_{eff} = \dfrac{2f}{(1 - e^{-\mu\alpha_{n-1}})f + 2e^{-\mu\alpha_{n-1}}}$.

For f = 1 one finds f_{eff} = 1.374.

The maximum force on the belay F_B^{max} can be easily expressed by the impact force F^{max} by eliminating all $k_i(x_i - x_{i-1})$ in (4)

$$F_B^{max} = k_1 \max(x_1 - x_0) = F^{max} \prod_{i=1}^{n-1} \rho_i^{-1} = F^{max} \cdot \exp\left(-\mu \sum_{i=1}^{n-1} \alpha_i\right)$$

The sum of all α_i appear in the exponent, thus for high friction the exponential leads to a very small F_B^{max}: the impact force is distributed among the protection points and cannot propagate to the belayer.

The maximum force on the last protection point (LPP) is given by

$$F_{LPP}^{max} = 2F^{max} - F_{n-1}^{max} = F^{max}(1 + e^{-\mu\alpha_{n-1}}).$$

With the values above, we obtain F_{LPP}^{max} = 1.46 F^{max}.
Although F^{max} of equation (9) is increased by dry friction, the factor $(1 + e^{-\mu\alpha_{n-1}})$ can overcompensate this effect and in total F_{LPP}^{max} can be lower for moderate friction.

Finally the fall factor for an entire climbing route is shown in Fig.3. We take case 6 from Fig.4 of the next section. The length of each rope segment is l_i = 2m. All ρ_i are the same as in Fig.4 except with a very high ρ_5=10 resulting in a high fall factor after the climber is beyond P_5. Note the typical climbing situation which is most dangerous at the beginning of the climb before the first protection is reached. After clipping the first protection point the fall factor jumps to zero, then increases again after climbing away from the protection.

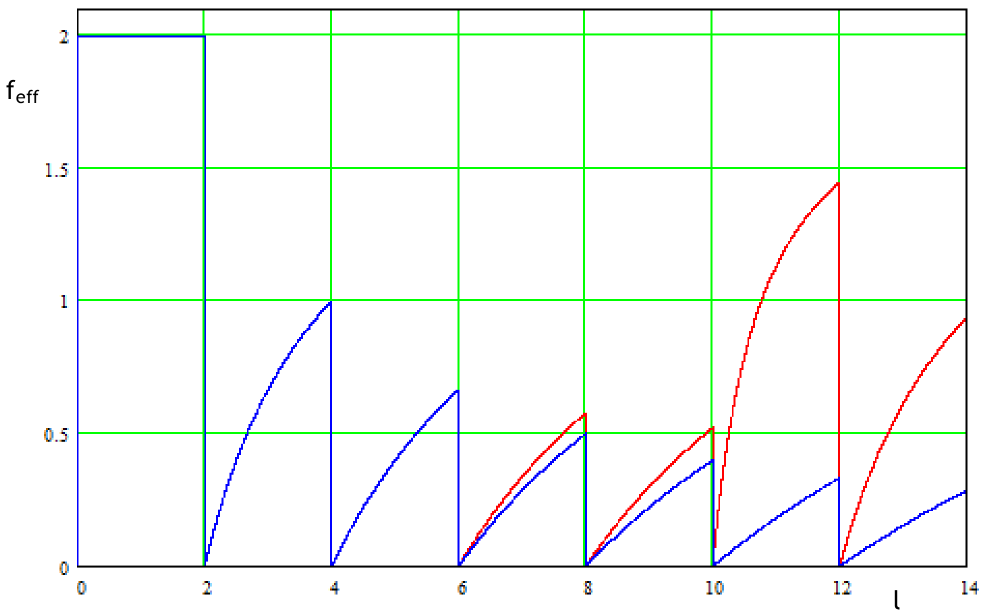

Fig.3: Fall factor as a function of rope length l of an entire climbing route with 6 protection points (case 6 of Fig.4). Blue: the fall factor without dry friction, red: with dry friction.

2. Rope drag

It is also possible to calculate the rope drag, i.e. the friction of the rope plus its weight that the climber feels when moving forward.
The force before P_1 (coming from below) is given by $\gamma l_1 \cos(\beta_1)$ (γ = density·g is the specific weight of the rope), so that the force after P_1, using the equation of Euler-Eytelwein, is given by

$$T_1 = \gamma l_1 \cos(\beta_1) e^{\mu\alpha_1}$$

which is larger than $\gamma l_1 \cos(\beta_1)$. The force after P_2 is the sum of T_1 and the weight of the next line element l_2 multiplied by $e^{\mu\alpha_2}$ in order to overcome the friction at P_2:

$$T_2 = (T_1 + \gamma l_2 \cos(\beta_2)) e^{\mu\alpha_2}$$

After P_i we have

$$T_i = (T_{i-1} + \gamma l_i \cos(\beta_i)) e^{\mu\alpha_i} = \gamma l_i \cos(\beta_i) e^{\mu\alpha_i} + \gamma l_{i-1} \cos(\beta_{i-1}) e^{\mu(\alpha_i + \alpha_{i-1})} + \ldots + \gamma l_1 \cos(\beta_1) e^{\mu(\alpha_i + \alpha_{i-1} + \ldots + \alpha_1)} \qquad (10)$$

Finally one arrives at the last P_{n-1}. The minimal drag force F_D that the climber needs to move forward is now given by

$$F_D = T_n = T_{n-1} + \gamma l_n = \gamma\left(l_n \cos(\beta_n) + l_{n-1} \cos(\beta_{n-1}) \cdot \rho_{n-1} + \ldots + l_1 \cos(\beta_1) \rho_1 \rho_2 \ldots \rho_{n-1}\right) \qquad (11)$$

or in a more compact notation defining an effective mass of the rope

$$F_D = g m_{eff}^{rope} = \gamma \sum_{i=1}^{n} l_i \cos(\beta_i) \prod_{j=i}^{n-1} \rho_j$$

In the case of no friction, if all ρ_i's are one, F_D is simply the weight of the rope γl multiplied by an average cosine: $< \cos(\beta) >= \frac{1}{l} \sum_{i=1}^{n} l_i \cos(\beta_i)$.

If the β_i, l_i and ρ_i are all constant, one obtains a drag force

$$F_D = \gamma l \cos(\beta) \frac{1}{n} \frac{e^{n\mu\alpha} - 1}{e^{\mu\alpha} - 1}$$

which increases exponentially with the total angle $n\alpha$. Under normal conditions when the α's are small, F_D is given by $F_D \cong \gamma l \cos(\beta)(1 + 1/2 \cdot \mu\alpha(n-1))$. For $\alpha = \pi/10$, $n = 10$ and $\mu = 1/4$ the effective weight the climber has to pull increases about 35% compared to the case without friction.

The next figure shows a rope with 6 protection points ρ_1 - ρ_6. The total angle deviation $4 \cdot \pi/4$ is the same in all 6 cases. In spite of the apparent equivalency, the rope drags are different. In case 1, only rope segment l_1 has to be pulled through ρ_1, ρ_2, ρ_3 and ρ_4, all larger than 1. In case 6, however, the segments l_1 - l_3 must be pulled through ρ_3 - ρ_6. Thus the rope drag is larger in this case, in contrast to intuition.

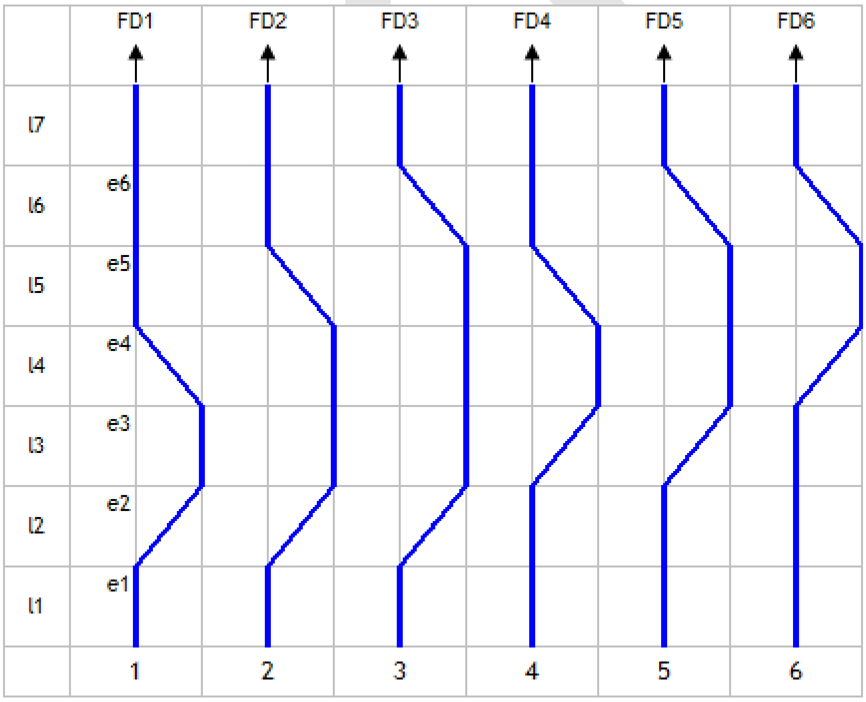

Fig.4: Six situations of rope drag with the same total friction angle. Case 1 has the lowest rope drag, the following are in ascending order ending with the highest rope drag for case 6. Taking for $\rho_i = \exp(\mu\pi/4) \approx 1.22$ one obtains in case 1 an effective mass $m_{eff}^{rope} = 1.38 m^{rope}$ and in case 6 $m_{eff}^{rope} = 1.73 m^{rope}$.

3. Conclusions

In this work we derived for the HO model with dry friction an expression for the impact force (equation 9) for all kinds of climbing situations: with arbitrary protection points, friction coefficients, angles between rope and protection points. It turned out that the original form of the equations is unchanged, if one redefines the spring constant of the rope by introducing an effective rope length, which leads to an effective fall factor. Because of the easy explicit expressions one can calculate at once the impact force for many climbing situations.

Dry friction leads first of all to a higher (stiffer) effective elastic modulus. Energy dissipation due to dry friction is smaller than the strong viscous damping which starts near the force maximum [1]: the reason why a rope has almost no oscillation is viscous damping and not dry friction. In the limit of infinite dry friction there is only energy dissipation from viscous friction.

Furthermore we calculated the rope drag a climber has to overcome in order to move forward. Its only source is dry friction. It can also be expressed by an effective mass which is larger than the mass of the rope that has to be pulled by the climber. This effective mass depends exponentially on the sum of the angles of the direction changes the climber has made. “Early errors” not using longer runners to reduce the angles α at the first protection points are less severe than “later errors” which is in contrast to intuition.

Appendix A

In this appendix we discuss the equations (4) for n = 2, i.e. for only one protection point P_1, in more detail. This special case is important, because the last protection has usually the largest friction ($\alpha = \pi$), and is therefore a limiting case of (4) when all α_i can be neglected except of the last one.
Assuming for the moment a small mass m_1 at P_1, one obtains from the Lagrangian

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2 + m_2gx_2 \qquad \text{(A1)}$$

the following equations of motion

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= F_1 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= m_2g \end{aligned} \qquad \text{(A2)}$$

The friction force F_1 is given by

$$\begin{aligned} F_1 &= -(1 - e^{-\alpha_1\mu}) \cdot \mathrm{sgn}(\dot{x}_1) \cdot \max(|k_2(x_2 - x_1)|, |k_1x_1|) = \\ &- (e^{\alpha_1\mu} - 1) \cdot \mathrm{sgn}(\dot{x}_1) \cdot \min(|k_2(x_2 - x_1)|, |k_1x_1|) \end{aligned} \qquad \text{(A3)}$$

with the sign function defined as

$$\mathrm{sgn}(a) = \begin{cases} -1 & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ 1 & \text{if } a > 0 \end{cases}$$

The frictional force at P_1 always has the opposite sign of the velocity at P1, its magnitude is independent of the velocity, but depends on the maximum of the two forces acting on either side of P_1.
Equations (A2) must be solved with the initial conditions $\dot{x}_2(0) = v_0$, $\dot{x}_1(0) = 0$, $x_1(0) = 0$, $x_2(0) = 0$. The initial velocity v_0 after a fall of $2l_n$ is usually sufficiently large so that m_2g can be neglected.
Let us discuss the time interval until the rope attains its first zero crossing, beginning with the first half $0 \le t \le \frac{T}{4}$

$$\frac{T}{4} = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{2}\sqrt{\frac{m_2}{k_2}}$$

This time interval ends when the rope reaches its maximum elongation and impact force and during this time interval the following relations are valid

$$k_2(x_2 - x_1) > k_1x_1 > 0 \text{ and } v_2 > v_1 > 0$$

Taking F_1 from (A3) one gets

$$m_1\ddot{x}_1 + k_1 x_1 - \frac{1}{\rho_1} \cdot k_2(x_2 - x_1) = 0$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{A4}$$

and for $m_1 \to 0$

$$x_1 = \frac{k_2}{\rho_1 k_1 + k_2} x_2$$
$$m_2\ddot{x}_2 + \frac{k_1 k_2 \rho_1}{k_1\rho_1 + k_2} x_2 = 0 \tag{A5}$$

(see Figure 5). The second time interval $\left(\frac{T}{4} < t \le \frac{T}{2}\right)$ ends when x_1 and x_2 have again their initial values:

$$\frac{T}{2} = \pi\sqrt{\frac{m_2}{k_2}}$$

At the beginning of this time interval the velocities are zero, but we still have $k_2(x_2 - x_1) > k_1 x_1 > 0$. The total energy rate is given by

$$\frac{dE}{dt} = -v_1(t)\big(k_2(x_2(t) - x_1(t)) - k_1 x_1(t)\big) \le 0$$

and can never increase. This can only be satisfied if $v_1 \ge 0$. Thus, it follows
$v_1 = 0$ at least as long as $k_2(x_2 - x_1) > k_1 x_1$. x_1 is constant and there is no energy dissipation. That is very surprising and interesting. One could expect that the motion of x_1 begins again at the time corresponding to $k_2(x_2 - x_1) = k_1 x_1$.
Numeric integration of the equations of motion, however, show that the time interval with $v_1 = 0$ ends at a time t_1 which is somewhat longer than the time corresponding to $k_2(x_2 - x_1) = k_1 x_1$. At the time t_1 the velocity v_1 immediately jumps to

$$v_1 = \frac{k_2}{\rho_1 k_1 + k_2} v_2$$

valid until the zero crossing of the rope elongation x_2.

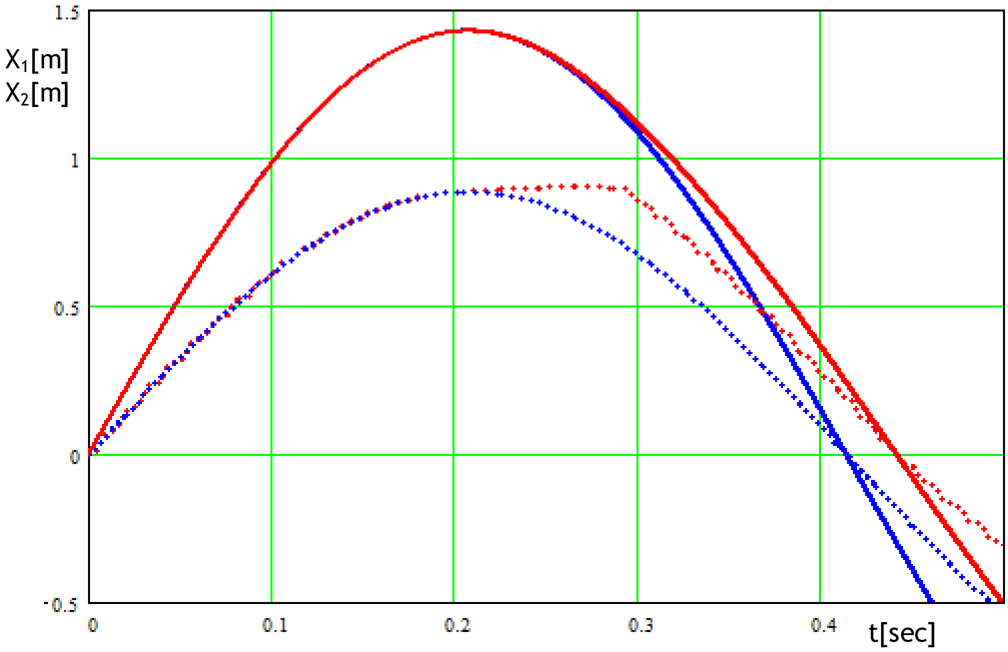

Fig.5: The blue curves are the elongations x_1 (solid) and x_2 (dotted) from equations (A5) as a function of time. The red ones are calculated numerically. Equations (A5) are exact up to the maximum of x_1 and x_2.

A full discussion of dry friction is beyond the level of this paper and fortunately not particularly important here, because we are interested in the influence of dry friction on the maximum elongation and impact force which takes place at time T/4.

