

The omnipresent impact force formula for a climbing rope

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This work demonstrates the omnipresence of the known impact force formula. Although originally derived only for the straight fall with a linear elastic rope, it applies almost unchanged for many other, more complex fall models and situations.



Typical straight fall in a klettergarten situation with a low fall factor [9].

In the following we will derive the well-known impact force formula as simply as possible and show its importance for more complex fall models and situations. It turns out that the same form of the impact force formula can also describe falls with internal and external friction, with slack rope and under an oblique fall angle. It even appears in its original form in modeling rope brakes of belay devices.

The present work addresses the interested non-specialist who has some knowledge in physics, such as the conservation of energy, and maybe even has a rudimentary understanding of Newton's equations of motion. Since nowadays knowledge gaps can be easily filled with help of the Internet, a certain mathematical level of the reader can be presupposed. While the mathematical requirements are relatively low, the possibilities to perform own calculations and estimates with the simple results and formulas are large.

In this article many interesting and important practical questions will be answered. What are the forces on the climber at a certain height of fall? Is slack rope of advantage or disadvantage? What effect has the friction between rope and carabiner on the climber, the protection points and the belayer? Are fall factors larger two possible in climbing situations? How can a belay device (rope brake) be modeled? What happens when you fall in a traverse?

The impact force formula dates back to Arnold Wexler who in 1950 was the first to apply it to the belaying in climbing [1]. Of more recent times are some Internet articles on fall physics which are also kept simple. Worth reading are [2,3,4,8], others such as [5,6] must be read with caution, because some errors have crept in.

Derivation and discussion of the impact force formula

To derive the known impact force formula one first needs a material equation that describes the relation between the rope tension and the rope stretch. For not too high stretches this relation is linear and, in the nomenclature of elasticity theory, given by Hooke's law

$$\sigma = E \cdot \varepsilon \quad (1)$$

with the stress $\sigma = \frac{\text{force } F}{\text{rope cross section } A}$, the strain $\varepsilon = \frac{\text{elongation } y}{\text{rope length } L}$ and the elasticity modulus E as a material constant. The larger E the more stress is needed to achieve a certain strain. The unit of E is the same as for stress, i.e. $\text{N/m}^2 = 1 \text{ Pa}$. The force $F = \sigma A$ is given by:

$$F = \left(\frac{EA}{L} \right) y = ky \quad (2)$$

The term EA/L is the spring constant k which depends on L . This is important for the following and also easy to understand: a long rope is easier to stretch by the same y than a short rope.

Now you need only energy conservation to express the initial velocity (where the rope begins to act) as a function of the fall height h : $1/2 mv_0^2 = mgh$ leads to $v_0 = \sqrt{2gh}$.

At the turning point where the stretch has its maximum y_{\max} , the restoring force is maximum F_{\max} and the velocity is zero, the conservation of energy reads (first without slack rope, that is $s = 0$)

$$mgh = -mgy_{\max} + \int_0^{y_{\max}} F(y) dy \quad (3)$$

The gravity part of the potential energy is reduced by the elongation (the coordinate system is chosen such that y thereby increases, see Fig.1). The other part, the stress energy, would be simply $F \cdot y$ in the case of a constant force, i.e. force·elongation. For our elongation-dependent force, however, one has to add up the sections $F(y)dy$, i.e. calculate

the integral $\int_0^y F(y) dy$ which is easy to calculate for the linear y -dependence of F .

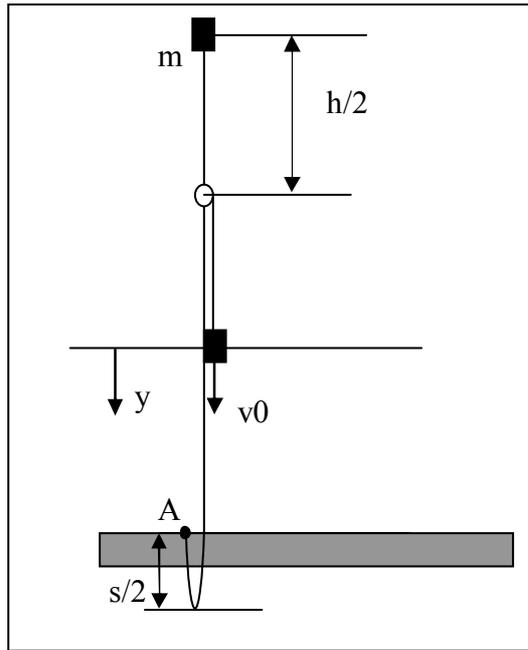


Fig. 1: A straight fall where the mass m falls a distance h and has the velocity v_0 at $y=0$. At this point the rope (with length L and fixed at A) begins to stretch when there is no rope slack ($s=0$).

The result is

$$mgh = -mgy_{\max} + \frac{EA}{L} \frac{y_{\max}^2}{2} \quad (4)$$

which can be rewritten as $mgEA \frac{h}{L} = -mgF_{\max} + \frac{1}{2} F_{\max}^2$ with the use of $y_{\max} = \left(\frac{L}{EA} \right) \cdot F_{\max}$.

The ratio fall height to working rope length is called fall factor

$$f = h/L \quad (5)$$

Solving for F_{\max} one gets

$$F_{\max} = mg + \sqrt{2mgEaf + m^2g^2} \quad (6)$$

F_{\max} is called impact force. If F_{\max} exceeds a critical value, the rope will break.

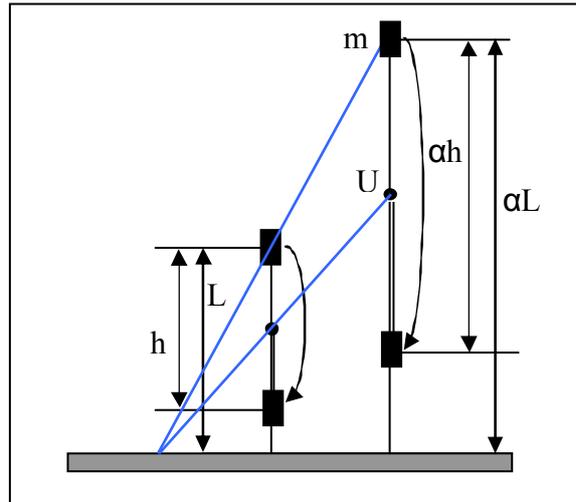


Fig. 2: Scale invariance of the impact force:
All falls, where the mass m and the position of the last runner U lie on arbitrary superposed points on the blue lines, have the same fall factor and thus the same impact force.

In F_{\max} the fall height h appears only as the ratio h/L , therefore F_{\max} depends only on the fall factor f . This is an important result. It provides a scaling law (see Fig.2), because F_{\max} remains constant when rescaling the fall height h and the length L of the paid-out rope by the same factor. It is amazing that a fall from a small fall height and $f = 2$ (i.e. a fall in the belay in a multi-pitch route) causes the same high forces as a fall from greater height h .

While in climbing the fall height h cannot exceed $2L$ and thus the fall factor is in the range $0 \leq f \leq 2$, this is different in via ferratas. There, L ($\approx 1\text{m}$) is the length of the lanyard which connects the climbers with the cable of the via ferrata. The fall height can be several meters depending on the distance between two adjacent rock anchors which may lead to fall factors larger 2.

For $2mgEaf \gg m^2g^2$ (i.e. for large falls such as the standard UIAA fall), then one can use a simple equation for F_{\max} and y_{\max} :

$$F_{\max} \approx m\omega v_0 = \sqrt{2mgEaf} \quad \text{und} \quad \frac{y_{\max}}{L} = \frac{F_{\max}}{EA} \approx \sqrt{\frac{2mgf}{EA}}$$

What happens when E is reduced to one half (for example when one clips only one of the two twin ropes instead of clipping them together)? Then F_{\max} is smaller by $1/2\sqrt{2} \approx 0.71$, the impact force reduction therefore is only about 30%. The stretch y_{\max} , however, is larger by $\sqrt{2}$, the product $F_{\max} \cdot y_{\max}$ remains the same.

Next we estimate the modulus of elasticity of a climbing rope using the derived formulas. Solving equation (6) for EA one obtains $EA = 1/f(F_{\max}/2mg - 1) \cdot F_{\max}$. With $F_{\max} = 8\text{kN}$ as a typical value for the standard impact force $EA = 18.5\text{kN}$. Our oscillator model, however, is not quite correct, because the measured dynamic relative stretches y_{\max}/L of about 35% (in a standard UIAA fall) are smaller than those calculated from equation (2) $y_{\max}/L = F_{\max}/EA$ with about 43%. But in view of the simple model with its simplified assumptions the result is not too bad and therefore suitable for estimates.

For a climbing rope with a radius of 5mm one obtains for the elasticity modulus $E = 0.24\text{GPa}$. For a steel cable of the same diameter E is more than 10-fold, a hemp rope has an $E \approx 0.6\text{GPa}$, a rubber rope is between 0.001 and 0.01 GPa.

The largest possible F_{\max} (with $m = 80\text{kg}$ and the maximum possible fall factor $f=2$) is only 6% larger than the impact force in a standard UIAA fall. With a typical tensile strength of about 20kN, i.e. more than twice as much, a climbing rope cannot break by pure stretch in a normal fall (without a sharp edge to cut the rope). With hemp ropes this is different. The tensile strength of a hemp rope is approximately 8kN, which means that in a standard fall with impact force of around 14kN (using $E = 0.6\text{GPa}$) a hemp rope is long broken.

Although the equation of motion for the described fall was not needed to derive F_{\max} , we will specify it here, because equations of motion will be used later. The sum of all forces is equal to mass times acceleration, so that

$$m\ddot{y} = -ky + mg. \quad (7)$$

This is the equation of a harmonic oscillator with an external constant force. With the initial conditions $y(0)=0$ and $v(0)=v_0$ the solution is

$$y(t) = \frac{v_0}{\omega} \sin(\omega t) + \frac{g}{\omega^2} (1 - \cos(\omega t))$$

with $\omega = \sqrt{k/m}$. One thus obtains not only F_{\max} or y_{\max} , but also the full time behavior of the stretch, velocity, and acceleration.

Next, assume that the fall height h in the impact force formula (6) approaches zero. One obtains not, as one might think, the static limit $F_{\max}^{\text{stat}} = mg$, but the double, i.e. $F_{\max} = 2mg$, because the rope oscillates due to the initial $\ddot{y}(0) = g$.

In the static case, the relative rope stretch is given by

$$\frac{y_{\max}^{\text{stat}}}{L} = \frac{mg}{EA}$$

With an average static rope stretch of approximately 8% (averaged over many climbing ropes) and a mass $m = 80\text{kg}$, one obtains an $EA = 9.81\text{kN}$, much smaller than the above calculated value of EA . This means that one cannot adequately describe the rope with only one modulus of elasticity, as for longer times (= static case) the rope behaves much softer than for short periods.

We now discuss the forces involved. To an outside observer, the tension in the rope before the rope begins to act is zero, the force on the climber, however, is mg . The maximum rope stress is F_{\max}/A , the maximum force on the climber is $F_{\max} - mg$.

From the perspective of the climber (i.e. in his inertial frame of reference) he is weightless during his free fall. When the rope begins to act this brake force is added to the force of gravity. At the time of the maximum impact force the effective weight of the climber is F_{\max} which, under standard fall conditions, is 8 kN, i.e. about 10g. But the duration of this enormous force is only about a tenth of a second.

Impact force with slack rope

If $s > 0$ (see Fig. 1), either by an inattentive belayer or by extra paid-out rope when the climber is going to clip a carabiner, both the fall height h and the rope length L is increased by s . In equation (4) h is replaced by $h + s$ and L by $L + s$:

$$F_{\max} = mg + \sqrt{2mgEA \frac{h+s}{L+s} + m^2 g^2} . \quad (6')$$

That is so far trivial, but the result is interesting. For this purpose we transform $\frac{h+s}{L+s}$ into

$\frac{h+s}{L+s} = f = f_0 + (1-f_0) \frac{s}{L+s}$, where $f(s=0) = f_0 = h/L$. For $f_0 > 1$, possible in multi-pitch routes, the second term of the above equation is negative and reduces the fall factor f with respect to no slack. At least theoretically, slack rope is advantageous when you have to climb far above the belay or the first bolt and, in a fall, you end up below the belayer. But only if it is guaranteed that the increased fall height does not lead to a greater impact on the rock or a ledge or even a collision with the belayer. For $f_0 < 1$, however, the fall factor is increased. Since this is the normal case (always in the klettergarten when high external friction can be neglected (see next section)), slack rope leads to harder falls and must therefore be avoided.

In top-rope climbing the fall height is zero. The maximum force on the rope is obtained by setting $h = 0$ in equation (6'). For $s = 0$ in addition, one obtains for F_{\max} the smallest possible value of $2mg$. Thus the belayer must hold twice the bodyweight in a fall into the top-rope without slack. The force on the top anchor is therefore four times the weight of the climber. Surprised?

Impact force with external (dry) friction

In order to consider the external friction (covered in more detail in [7]), we must use the equation of motion for the fall mass m . In a first step, it is set up without friction at the position of the last bolt U . There is equilibrium of the spring forces, namely

$$k_2(y_2 - y_1) = k_1 y_1$$

(see Fig.3). $k_1 = k \frac{L}{L-h/2}$ and $k_2 = k \frac{L}{h/2}$ satisfy the relation $1/k = 1/k_1 + 1/k_2$ of spring constants in series.

Therefore the overall spring force ky_2 can also be written as

$$ky_2 = \frac{k_1 k_2}{k_1 + k_2} y_2 = k_1 y_1 = k_2 (y_2 - y_1),$$

so that the following equation is equivalent to (7):

$$m\ddot{y}_2 + k_2(y_2 - y_1) = mg . \quad (8)$$

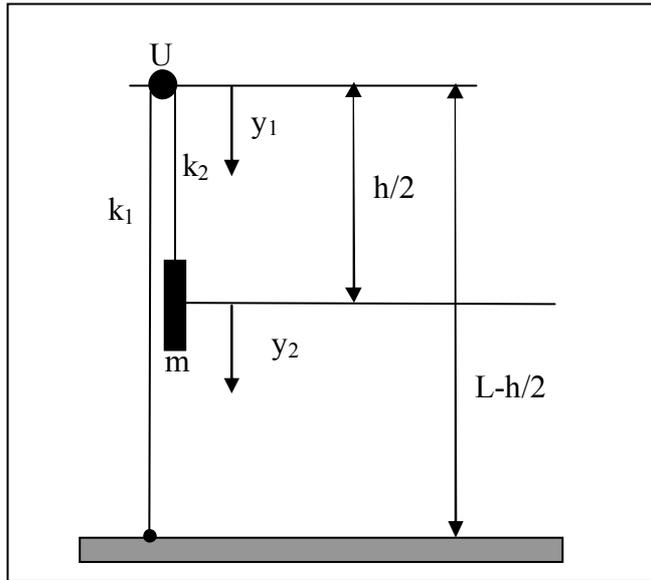


Fig.3: The same fall situation as in Fig.1 with additional spring constants k_1 and k_2 left and right of the deflection U. y_1 is the shift of a small rope element under an applied force with U as reference point.

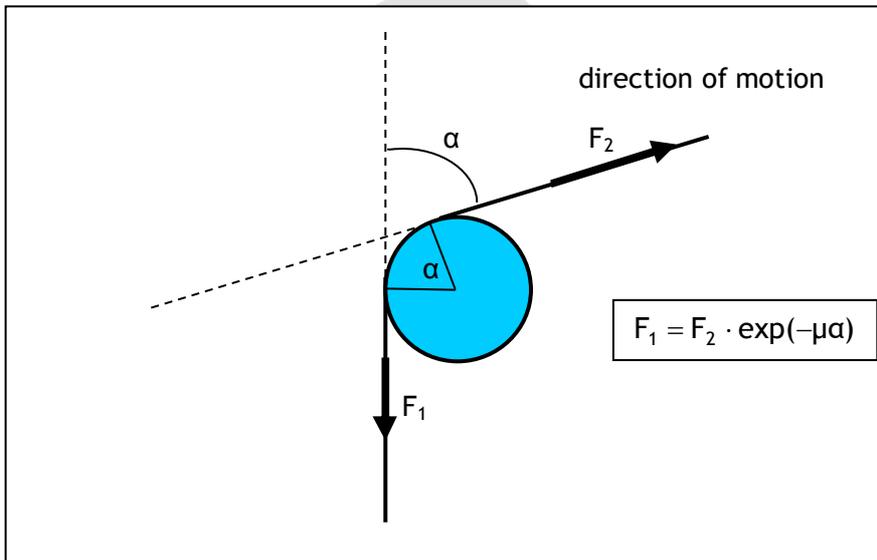


Fig.4: A rope is pulled over a curved surface with a coefficient of friction μ . When the pulling force is F_2 , then the reactive force is reduced and given by $F_1 = F_2 \exp(-\mu\alpha)$. This has important implications in abseiling. There, F_2 is the weight of the abseiling person. The force F_1 that must be hold with his hands is reduced by about a factor $e^{\mu\alpha} \approx e^{2\pi \cdot 0.35} \approx 10$ (depending on the friction coefficient, i.e. new or old ropes). α is slightly more than 2π for a figure eight belay device.

To describe the friction we use the Euler-Eytelwein formula (see Fig. 4) which gives a relationship between the forces before and behind the deflection U. It changes the above equation for the equilibrium of forces into

$$k_1 y_1 = k_2 (y_2 - y_1) \frac{1}{\rho} \quad (9)$$

with $\rho = e^{\mu\alpha}$. ρ contains the friction coefficient μ and the angle α that the rope forms with the friction point ($\alpha \approx \pi$). Measured values of μ between steel and nylon are about 0.35. The force occurring on the left side of U (which equals the force that the belayer must hold) is reduced by the external friction by a factor $\rho^{-1} = e^{-\mu\alpha}$.

Solving the last equation (9) for y_1 and inserting it in (8), an equation of motion for y_2 alone is obtained

$$m\ddot{y}_2 + \frac{k_1 k_2 \rho}{\rho k_1 + k_2} y_2 = mg \quad (10)$$

(valid for $\dot{y}_2 \geq 0$). This is the same equation of motion as without friction (7), only with a new effective spring constant

$$k_{\text{eff}} = \frac{k_1 k_2 \rho}{\rho k_1 + k_2} = EA \frac{1}{(L - \frac{h}{2})\rho^{-1} + \frac{h}{2}} = EA \frac{1}{L_{\text{eff}}} \quad (11)$$

with $L_{\text{eff}} = (L - \frac{h}{2})\rho^{-1} + \frac{h}{2}$. As expected, $k_{\text{eff}} \geq k = \frac{k_1 k_2}{k_1 + k_2}$ (the limit without friction).

Since the initial conditions have not changed and the equations of motion (7) and (9) differ only by the spring constant, F_{max} is given immediately in analogy to (2):

$$F_{\text{max}} = mg + \sqrt{2mghk_{\text{eff}} + m^2g^2} = mg + \sqrt{2mgEAf_{\text{eff}} + m^2g^2} \quad (6'')$$

with an effective fall factor

$$f_{\text{eff}} = \frac{h}{L_{\text{eff}}} = \frac{\rho f}{1 + (\rho - 1)f/2} \quad (11)$$

For zero friction, that is $\rho = 1$, we get $L_{\text{eff}} = L$ and $f_{\text{eff}} = f$. For increasing friction L_{eff} becomes smaller and in the limit of large friction, that is $\rho \gg 1$, L_{eff} approaches $h/2$. Then a fall factor of 2 is obtained. The strong friction acts as if the rope is fixed at the last bolt. So even for paid-out ropes with a large L a fall factor near 2 is possible. If there is additional friction against the rock, it may even get worse. Consider the fall into a bolt placed above a roof and the rope strongly rubbing against the roof edge, then L_{eff} is the length of rope between the falling climber and the roof edge with a possible f_{eff} greater than two (see Fig. 5).

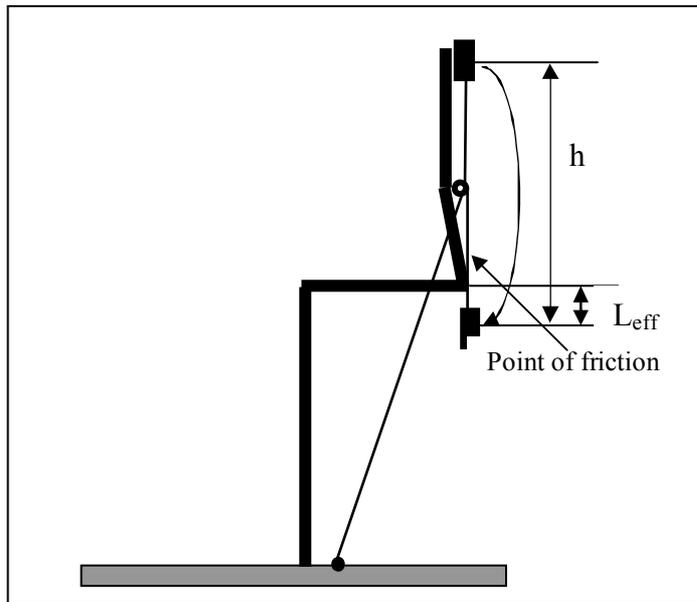


Fig.5: Fall over a roof edge. With external friction fall factors greater than 2 are possible, because for a fall height h L_{eff} can become smaller than $h/2$.

Force on the last clipped bolt and on the belayer

The force F^U on the last anchor U is the sum of the two spring forces left ($k_1 y_1$) and right ($k_2(y_2 - y_1)$) of U:

$$F^U = k_2(y_2 - y_1) + k_1 y_1 = \left(1 + \frac{1}{\rho}\right) k_2(y_2 - y_1) = \left(1 + \frac{1}{\rho}\right) k_{\text{eff}} y_2 \quad (12)$$

Hence the maximum of F^U is $F_{\text{max}}^U = \left(1 + \frac{1}{\rho}\right) F_{\text{max}}$ with the limiting cases

$$F_{\text{max}}^U = \begin{cases} 2F_{\text{max}} & \rho = 1 \\ mg + \sqrt{4mgEA + m^2 g^2} & \rho \gg 1 \end{cases}$$

Using the standard UIAA fall as a measure, then F^U without friction is about 16 kN. This is quite worrisome, because it is not too far away from 25kN, the standard holding force for bolts (in radial direction).

What is the maximum force that the belayer must hold? As equally large reaction force to $k_1 y_1$ it is given by

$$F_{\text{max}}^S = k_1 y_1 = k_{\text{eff}} y_2 \frac{1}{\rho} = \frac{1}{\rho} F_{\text{max}} \quad (13)$$

For large friction the force on the belayer goes to zero (see Fig. 6).

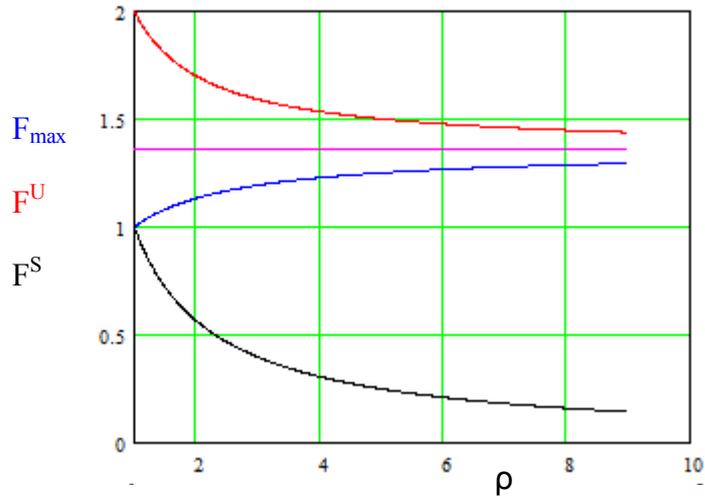


Fig.6: While the external friction ρ makes the fall F_{\max} harder for the falling climber (blue curve), the force F^S on the belayer (black) and the force F^U on the last clipped bolt (red) become smaller. The straight line (magenta) is the limiting case of fall factor $f = 2$. All falls are normalized to F_{\max} without friction.

Energy balance at the last clipped bolt in the presence of external friction

We still have the energies occurring at the maxima of force resp. stretch. The elastic energy $E_{\text{dehn}} = \frac{k_2}{2}(y_2 - y_1)^2 + \frac{k_1}{2}(y_1)^2$ is given by

$$E_{\text{dehn}} = \frac{1}{2}k_1k_2 \frac{k_2 + \rho^2k_1}{(k_2 + \rho k_1)^2} y_{2\max}^2 = \frac{EA}{L} y_{2\max}^2 \frac{2 + (\rho^2 - 1)f}{2 + (\rho - 1)f} \quad (14)$$

after the elimination of y_1 by means of eq.(9). With the simplified form

$$\frac{y_{2\max}}{L_{\text{eff}}} = \frac{F_{\max}}{EA} \approx \sqrt{\frac{2mgf_{\text{eff}}}{EA}} \quad \text{one gets}$$

$$\frac{E_{\text{dehn}}}{mgh} = \frac{1}{\rho} \cdot \frac{2 + (\rho^2 - 1)f}{2 + (\rho - 1)f} \quad (14')$$

The total energy E_{total} is given by the parts E_{dehn} , the dissipated energy E_{diss} which is converted into heat (up to the maximum stretch), and the potential energy E_{pot}

$$E_{\text{total}} = E_{\text{dehn}} + E_{\text{diss}} + E_{\text{pot}} = \frac{1}{2} y_{2\max}^2 \left(\frac{k_1k_2\rho}{k_2 + k_1\rho} \right) - mgy_{2\max} = \frac{EA}{L} \frac{\rho}{2 + f(\rho - 1)} y_{2\max}^2 - mgy_{2\max} \quad (15)$$

Now E_{diss} can be determined

$$E_{\text{diss}} = \frac{EA}{L} y_{2\text{max}}^2 \frac{(2-f)(\rho-1)}{(2+f(\rho-1))^2} \quad (16)$$

and with the simplified form for $y_{2\text{max}}$

$$\frac{E_{\text{diss}}}{mgh} = \frac{1}{\rho} \frac{(2-f)(\rho-1)}{2+f(\rho-1)} \quad (16')$$

E_{diss} disappears for $\rho = 1$ (no friction) as well as in the case of large friction $\rho \gg 1$, so in between there is a maximum. E_{diss} disappears also for $f = 2$. This is the reason why in a first approximation the external friction can be neglected in the discussion of the standard UIAA fall. For small f , however, E_{diss} may be greater than E_{dehn} .

Impact force of a rope with internal viscous friction

The known Maxwell model (see for example [7]) can be used to approximate the internal friction of a rope. Microscopically internal friction arises, because the polymer molecules of the threads making up a rope do not react instantaneously to mechanical stress but only delayed.

The model consists of an elastic element in series with a viscous damping element (Fig.7).

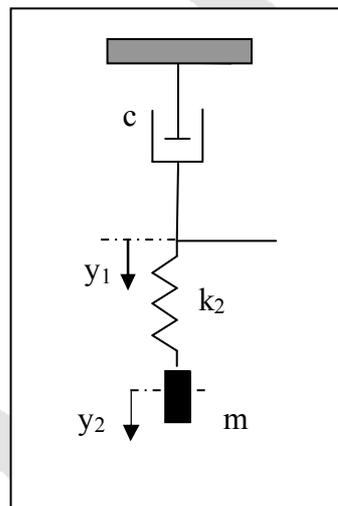


Fig.7: Maxwell model with a spring k_2 and a viscous element c .

The corresponding equations of motion are easily derived: the forces on the mass m with the coordinate y_2 are gravity and the restoring force of a spring $k(y_2 - y_1)$ in analogy to (8), so that Newton's equation of motion is given by

$$m\ddot{y}_2 + k(y_2 - y_1) = mg \quad (17)$$

A second equation for y_1 describes the damping element with a friction force $c\dot{y}_1$. This force counteracts the spring force which leads to

$$k(y_2 - y_1) - c\dot{y}_1 = 0 \quad (18)$$

In order to not unnecessarily complicate the calculations, we only give the exact results of the above two differential equations for $g = 0$. The initial velocity is as before $v_0 = \sqrt{2gh}$. Setting g to zero means that from the moment when the rope begins to stretch, g is switched off. This is a very good approximation for larger falls.

The solutions for F and for the internal coordinate at an initial speed v_0 are

$$F(t) = k_2(y_2(t) - y_1(t)) = m \frac{\omega^2}{\Omega} v_0 e^{-\kappa t} \sin(\Omega t) \quad (19)$$

$$y_1(t) = \frac{2\kappa v_0}{\omega^2} \left(1 + e^{-\kappa t} \left(\cos(\Omega t) + \frac{\kappa}{\Omega} \sin(\Omega t) \right) \right)$$

$$\text{with } \Omega = \sqrt{\omega^2 - \kappa^2}, \quad \omega^2 = \frac{k}{m} \quad \text{and} \quad \kappa = \frac{k}{2c}.$$

The maximum force is given by

$$F_{\max} = m\omega v_0 \exp\left(-\frac{\kappa}{\Omega} \arctan\left(\frac{\Omega}{\kappa}\right)\right) \approx m\omega v_0 \exp\left(-\frac{\pi\kappa}{2\Omega} + \frac{\kappa^2}{\Omega^2}\right)$$

This has to be compared with the former result $F_{\max} = m\omega v_0$ which was valid for $g=0$ and $\kappa=0$.

To establish an equivalent form to the known impact force formula, one can also write to a good approximation (the error is of the order κ^2/ω^2)

$$F_{\max} \cong mg + \sqrt{2mgEA \frac{h}{L} e^{-\frac{\pi}{2Q}} + m^2 g^2} \quad (6''')$$

$$\text{with } Q = \frac{\omega}{2\kappa}.$$

Q often is called quality factor. It is defined as the ratio of total energy and energy loss per oscillation period, therefore $1/Q$ is a measure for the damping. The internal friction decreases the impact force exponentially. For very large c , i.e. for $Q \gg 1$, the exponential term approaches 1 and the original equation (6) is regained.

Unfortunately it is not possible to determine Q and EA from F_{\max} and y_{\max} in an easy way. A typical value of Q for a climbing rope is $Q \sim 2$, that is the energy absorption in the Maxwell model is about $e^{-\frac{\pi}{8}} \approx 66\%$ up to the time of maximum impact force. But more detailed calculations show that the Maxwell model provides somewhat too high values for the energy absorption up to the maximum impact force.

The Maxwell model as a dynamic belay device

The results of the previous section can also be used to describe a belay device, if the viscous element in the Maxwell model is interpreted as external rope brake (HMS, tube,

etc.). c is now a control variable for the belayer with his belay device. The braking distance is given by

$$y_1(\infty) = S = \frac{2\kappa v_0}{\omega^2} = \frac{mv_0}{c}$$

(see eq. (19)). For large c $y_1(\infty)$ approaches zero, i.e. there is no rope passage and the old equation of motion (7) with the impact force (6) is regained.

When c is kept constant, S increases with the square root of the fall energy. The damping ratio for small κ , i.e. small braking distances S , is given by

$$\frac{F_{\max}}{F_{\max}(\kappa=0)} \approx \exp\left(-\frac{\pi\kappa}{2\omega}\right) = \exp\left(-\frac{\pi}{4} \frac{\omega S}{v_0}\right) = \exp\left(-\frac{\pi}{4} \frac{S\sqrt{2EAfgm}}{hgm}\right) = \exp\left(-\frac{\pi}{4} \frac{S}{h} \frac{F_{\max}^0}{mg}\right) \quad (20)$$

Thus the braking distance reduces the impact force exponentially. If f is kept constant, then also the ratio S/L must be kept constant, so that the damping factor (20) remains unchanged.

A numerical example with $F_{\max}^0/mg = 6$ and $S/h = 0.5m/5m$ leads to an impact force reduction of about 40%.

The energy E_A absorbed by the belay device can be determined by means of F_{\max} and is approximately given by $E_A = \pi/4 S \cdot F_{\max}$, thus braking distance times rope tension.

Extensions of this model for a belay device can be performed with good accuracy by using equation (17), wherein the variation with time of y_1 as a control function can be arbitrarily chosen. For example, an exact solution of (17) with $y_1(t) = S(1 - e^{-t/\tau})$ is possible with two control parameters S and τ .

To obtain an impact force reduction, obviously S must be positive. However, also $S < 0$ is possible, as the author once had to experience. If an overzealous belayer accelerates in the opposite direction during a fall, then the climber endures a much larger impact force than the one from equation (6).

If a deformation of the falling body is possible, then this can be taken as another energy absorbing process with an additional viscous element c_κ . The new composite viscosity constant is obtained by summing the reciprocals of c and c_κ .

Impact force in a pendulum fall

When the author started to work on this topic, he was not fully aware of how complicated the physics of the pendulum fall would become. The equations of motion for an elastic pendulum are non-linear with little chances to get simple results. Therefore, only the cases of small angle deviations from the straight fall and of transverses with an initial angle of 90° will be discussed.

Small angle

As in equation (3) we use the law of conservation of energy at the maximum stretch point of the rope (i.e. $\dot{y} = 0$, at angle θ_1). This point is not, as one might have guessed, exactly

$$F = ky \leq \sqrt{kmv_0^2} = F(\theta_0 = 0) \quad (24)$$

For not too large angles ($\theta_0 \leq 30^\circ$) the impact force is always less than or equal to the impact force in a straight fall with the same fall height. Therefore small angular deviations from the straight fall are not important for the force on the rope and the climber. Only the mere height above the last protection point matters.

Impact force in traverse falls



Fig.9: long traverse in alpine climbing.

Again, the maximum rope stretch y_{\max} occurs approximately at $\theta_1 = 0$. We therefore discuss the law of conservation of energy at this point. Since at the beginning of the fall in a horizontal traverse the initial fall height and thus the kinetic energy are both zero, one obtains

$$0 = -mg(y_{\max} + a) + \frac{1}{2}ky_{\max}^2 + \frac{1}{2}m(y_{\max} + a)^2\dot{\theta}_0^2 \quad (25)$$

Without solving the complicated equations of motion for y and θ , there is no way to determine $\dot{\theta}_0^2$. Therefore, we discuss two limiting cases.

Small E (soft ropes)

In this case one can neglect the rotational energy, because the rope is not forced to follow a curved path and initially stretches straight downwards. With (25) one obtains

$$-g(y_{\max} + a) + \frac{1}{2} \frac{EA}{Lm} y_{\max}^2 = 0$$

which, after solving for y_{\max} , yields the impact force

$$F_{\max} = ky_{\max} = mg + \sqrt{2EAmg\hat{f} + m^2g^2} \geq 2mg \quad (6''')$$

i.e. the famous impact force formula with a fall factor $\hat{f} = a/L$. It is $0 \leq \hat{f} \leq 1$ and for $E = 0$ the smallest value $2mg$ is obtained.

Large E (hard ropes)

We consider the equilibrium of forces at $\theta_1 = 0$. For large E the stretch resp. force maximum is precisely at this point. If the rope is infinitely stiff, then the rope tension ky_{\max} (which remains finite in the limit $k \rightarrow \infty$ and $y_{\max} \rightarrow 0$) is the counterforce to the weight mg and the centrifugal force $ma\dot{\theta}_0^2$. If the rope is elastic, then there is an additional negative acceleration term at this point which shortens the rope at the bottom and reduces the rope tension. This leads to

$$ky_{\max} \leq mg + ma\dot{\theta}_0^2$$

or $\dot{\theta}_0^2 \leq \frac{\omega^2 y_{\max} - g}{a}$ after solving for $\dot{\theta}_0^2$. Inserting $\dot{\theta}_0^2$ into eq. (25), the law of conservation of energy, along with $y_{\max} = 0$ leads to

$$0 = -mga + \frac{1}{2} ma^2 \dot{\theta}_0^2$$

and one finally gets

$$ky_{\max} = F_{\max} \leq 3mg.$$

The force on the climber and the rope tension in traverse falls are therefore between

$$2mg \leq F_{\max} \leq 3mg$$

while in the straight fall F_{\max} depends on E and can become arbitrarily large. Note again that a fall in a traverse is rather harmless with respect to the impact force, but due to the large area that will be swept by the rope and the falling climber there is a high risk to hit the rock with high rotational speed.

Conclusion

This paper discussed the impact force formula which appears in many variations in the physics of climbing ropes and falling masses.

But one should not forget that it is based on a very simple rope model. A very accurate description of the standard impact force, however, is possible only by considering the nonlinear rope behavior at larger stretches and the rope mass.

Also the exemplary nature of the assumed fall situations does not always entirely match the climbing practice. Some real-life situations are difficult or even impossible to be described mathematically, such as when the climber hits the rock. Often this is much more painful for the climber than the force exerted by the rope.

Despite all this the derived relations lead to a better understanding of all the forces that occur during a fall. In addition, the physics of falling as a field of applied mechanics is a lot of fun, not least because some of the answers are astonishing and do not comply with the intuition.

References

- [1] A.Wexler, *The Theory of Belaying*, reprinted from THE AMERICAN ALPINE JOURNAL Vol.VII, 4 (1950)
- [2] S.Attaway, *Rope System Analysis* (1996), lamountaineers.org/pdf/xRopes.pdf
- [3] R.Goldstone, *The Standard Equation For Impact Force* (2006), www.rockclimbing.com/cgi-bin/forum/gforum.cgi?do=post_attachment;postatt_id=746
- [4] T.Czermin, P.Dullnig, L.Mathelitsch, W.Schneider, *Bergsteigen und Klettern - was sagt die Physik dazu?*, Physik in unserer Zeit 2 (2001)
- [5] R.Stör, *Ungeahnte Kräfte*, Klettern Mai 2002
- [6] W.Fimml, M.Larcher, *Energie ist Kraft mal Weg*, Berg&Steigen 4 (2000)
- [7] U.Leuthäusser, *Physics of Climbing Ropes, Parts 1, 2 and 3* (2012), www.sigmadewe.com/bergsportphysik.html?&L=1
- [8] R.A.Smith, *The development of equipment to reduce risk in rock climbing*, Sports Engineering 1, 27-39 (1998), personal.strath.ac.uk/andrew.mclaren/Smith.pdf
- [9] Photo of a falling climber, with kind permission of Urs Leuthäusser