About walking uphill:

time required, energy consumption and the zigzag transition

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A physical model for walking uphill is introduced. It is based on simple principles like the conservation of energy and a force dependent efficiency coefficient. Excellent agreement with experimental data was achieved.

Introduction

In this work we present a physical model for walking uphill which includes not only the route profile but also such important variables as the mass and the power of the mountain walker. Starting with an equation for the mechanical power, which the mountaineer must exert, in combination with the efficiency of the muscle using Hill's equation of muscle contraction, a relatively simple formula of the walking time can be derived.

This formula is compared with various other approaches. Among them is the purely empirical "Swiss formula" which is used to calculate the walking times for all Swiss hiking trails. As a side note, we mention that two articles in the German weekly newspaper DIE ZEIT were the original motivation for this work. In two successive articles two journalists and a mathematician tried in vain to get to the central point [1,2].

Finally we discuss the so-called zigzag transition which appears in all models and approaches that have a convex form for the walking time as a function of the gradient.

The power balance for walking uphill

In walking uphill, P is the mechanical power which you have to generate against a total force F which is composed of the component of gravity F_β = mg·sin(ß) and of the force component F_{hor}·cos(ß) for walking horizontally (see Figure 1). The force for walking horizontally has a dissipative character, since the center of mass moves periodically up and down without regaining the potential energy in the downward movement.

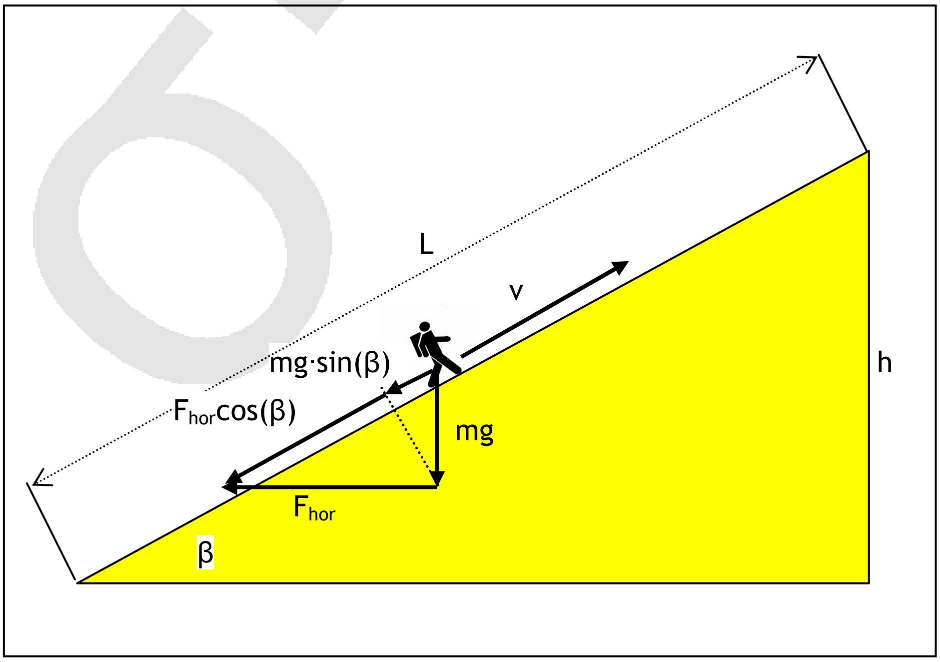

Fig.1

Thus the power balance is given by

$$P = (\vec{F}_{hor} + \vec{F}_{vert}) \cdot \vec{v} = F_{hor} v \cos(\beta) + mgv \sin(\beta) \tag{1}$$

It turns out, experimentally and also by models, that the force for walking horizontally varies linearly with the velocity, thus $F_{hor} = \lambda m v_{hor}$. Experiments show that the "friction constant" λ is about 4/3 sec^{-1} [4] and for the "inverted pendulum" model [5] one gets

$$\lambda_{IP} = \frac{\pi}{4}\sqrt{\frac{g}{6L_{leg}}} = 1.004 \cdot \sqrt{\frac{1}{L_{leg}}}\,.$$

With $F_\beta = mg\sin(\beta)$ this leads to

$$P = \lambda m v^2 \cos(\beta)^2 + F_\beta v \tag{2}$$

Force dependence of P and mechanical efficiency η

Quite obvious, the power P which is generated by muscles depends on the applied external force F. This can be seen from the fact that beyond a certain maximum load the muscle cannot work at all. The velocity of contraction becomes zero. Thus, one can not keep the power P constant by decreasing v up to an arbitrarily small value when F becomes very large which, in principle, is permitted by the equation P = Fv. Also, for too small forces the power decreases, because a rapid muscle contraction is associated with a small exerted force. This leads to a maximum of P at an optimal force F_{opt} which is associated with an optimum muscle velocity which for example in cycling determines the optimal pedaling frequency. Mathematically, this is expressed by Hill's equation of muscle contraction [6,7,8]. This equation gives a relation between the velocity of the muscle contraction and the applied external force F

$$P = F v_{max} \frac{F_{max} - F}{F_{max} + bF} \tag{3}$$

where v_{max} is the maximum velocity of contraction and F_{max} is the maximum (isometric) tension generated in the muscle. b is a parameter that lies between 1 and 4.

To keep the description as simple and general as possible, we consider the region around P_{opt} and expand the external force F at the optimal force F_{opt} that belongs to P_{opt} (see Figure 2)

$$P \cong P_{opt} - v_{max} \frac{(F - F_{opt})^2}{F_{max}\sqrt{1+b}}$$

Thus, using $P = \eta P_{opt}$, one can define a kind of mechanical efficiency η:

$$\eta = 1 - v_{max} \frac{(F - F_{opt})^2}{F_{max}\sqrt{1+b}} \frac{1}{P_{opt}} \cong \frac{1}{1 + v_{max} \frac{(F - F_{opt})^2}{F_{max}\sqrt{1+b}} \frac{1}{P_{opt}}} = \frac{1}{1 + a\left(\frac{F - F_{opt}}{mg}\right)^2} \quad (4)$$

Although η was derived by means of Hill's equation, it is more general, since η already follows from the fact that P has a maximum. η should not be confused with the energy conversion efficiency that describes the conversion of chemical energy into mechanical energy, which is about ¼. In the following we will use the last expression of (4) in order to avoid that the mechanical efficiency becomes negative. a can be expressed by Hill's parameters F_{max} and b and is given by

$$a = (mg)^2 \frac{v_{max}}{F_{max}\sqrt{1+b}} \frac{1}{P_{opt}} = \left(\frac{mg}{F_{max}}\right)^2 \frac{b^2}{\sqrt{1+b}(2+b) - 2(1+b)} \cong 4.7 \cdot \left(\frac{mg}{F_{max}}\right)^2 \quad (5)$$

The last relation is true for b = 4.

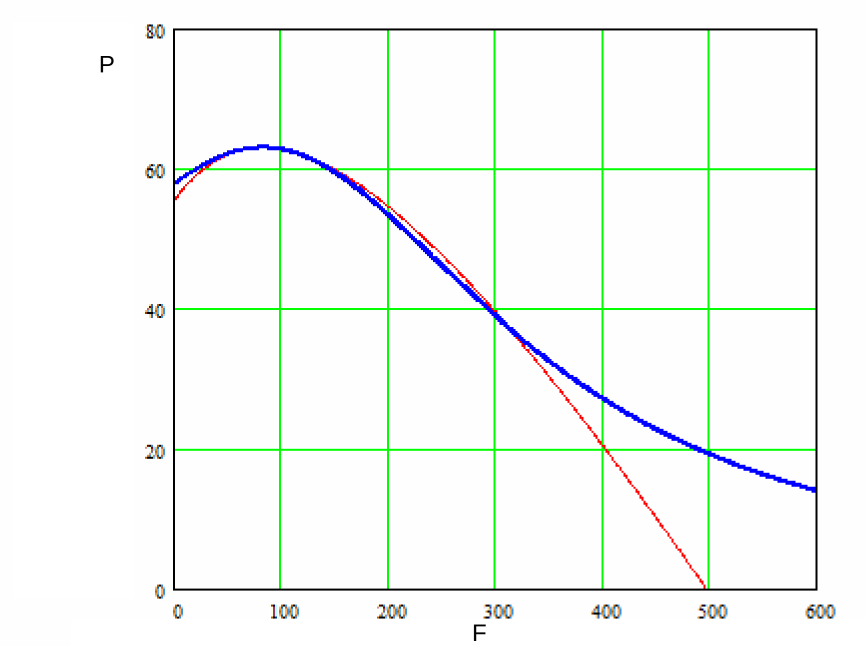

Fig.2: Mechanical power from Hill's muscle model (red curve) and the approximation ηP_{opt} from eq. (4) (blue curve) as a function of the external force F

Inserting η in (2) leads to $\dfrac{P_{opt}}{1+a\left(\dfrac{F-F_{opt}}{mg}\right)^2} = \lambda m v^2 \cos(\beta)^2 + F_\beta v$.

Since for larger β the velocity decreases sharply anyway, cos(β) is not particularly important. For simplicity, we set cos(β) = 1 and additionally assume that $F_{hor} \approx F_{opt}$. Although F_{opt} is slightly larger than F_{hor} for F_{max}~ mg, the error you make by this assumption is small (see the flat maximum of P in Figure 2). One obtains

$$\frac{P_{opt}}{1+a\left(\dfrac{F_\beta}{mg}\right)^2} = \lambda m v^2 + F_\beta v \qquad (6)$$

For β = 0 it follows $P_{opt} = \lambda m v^2$, the power equation for walking on flat ground. Going uphill with the bike gives a similar equation as (6), if you firstly take into account that, except at very steep gradients, one always achieves the efficiency η = 1 (i.e. a = 0) by choosing the optimal gear, and secondly you have to add the air resistance as an additional force (multiplied by v) on the right hand side of (6).

If the expression (6) is expressed by h, L and T, it follows

$$\eta(h/L)\cdot P_{opt} = \frac{P_{opt}}{1+a\left(\frac{h}{L}\right)^2} == \lambda m\left(\frac{L}{T}\right)^2 + mg\frac{h}{T} \tag{7}$$

Solving for T one obtains

$$T(h,L) = \frac{1}{2\cdot\eta(h/L)\cdot P_{opt}}\left[mgh + \sqrt{(mgh)^2 + 4P_{opt}\eta(h/L)\lambda mL^2}\right] \tag{8}$$

It turns out that, to a very good approximation, the efficiency η(h/L) under the root can be set to one so that (8) becomes even simpler:

$$T(h,L) = \frac{L}{2\cdot\eta(h/L)\cdot\left(\frac{P_{opt}}{m}\right)}\left[g\frac{h}{L} + \sqrt{\left(g\frac{h}{L}\right)^2 + 4\left(\frac{P_{opt}}{m}\right)\lambda}\right] \tag{9}$$

For h = 0 you get the walking time on flat ground which is given by $T(0,L) = L\sqrt{\frac{\lambda m}{P_{opt}}}$. The friction constant λ of the internal friction leads to longer walking times. Probably also the condition of the ground, which is an important factor while walking in the mountains, could be included in this friction constant as an external friction. T(h,L) at constant h/L is proportional to the walking distance L. Thus, keeping the gradient constant and doubling the walking distance L, leads to a doubling of the walking time. This certainly applies only in a specific time range, because fatigue effects must be considered for very short as well as very long ascents. In practice, constant speeds can be maintained over relatively long periods, if sufficient rests are taken, so that the above proportionality is valid.

We now compare the walking time (9) with three other approaches from literature.

1. The oldest relationship is a rule of thumb by Naismith which in its simplest form says: Allow 1 hour for every 4.8 km (originally 3 miles) forward, plus 1 hour for every 600 meters of ascent. Summarized in a formula, this means:

$$T = \frac{L}{v} + \frac{h}{v_{vert}} \tag{11}$$

with

$$v_{vert} = 0.6\frac{km}{h} = 0.17\frac{m}{sec} \quad \text{und} \quad v = 4.8\frac{km}{h} = 1.33\frac{m}{sec}.$$

If one assumes a velocity independent force along the way and includes the gravitational force, then it follows in analogy to (1), $P = \vec{F} \cdot \vec{v} = vF_0 + vmg\sin(\beta)$, and solved for T

$$T = \frac{L}{P}(F_0 + mg\sin(\beta)) = \frac{LF_0}{P} + \frac{mg}{P}h \tag{12}$$

By comparison with (11) it follows $v_{vert} = \frac{P}{mg}$. With the default values P=125W and m=70kg one obtains v_{vert} = 0.18 m/sec which is close to Naismith' value of 0.17 m/sec.

2. A simple exponential relationship is derived by Davey, Hayes and Norman [9], abbreviated herein as DHN, which is used in the work of Kay [10]. It is given by

$v_{hor} = \frac{L_{hor}(h)}{T} = v_{hor}(\beta = 0)e^{k\cdot\beta}$, so that

$$T_{DHN} = \frac{L_{hor}(h)}{v_{hor}(\beta = 0)}e^{-k\beta} = T_{DHN}(\beta = 0)\cos(\beta)e^{-k\beta} \tag{13}$$

which is only valid for gradients greater than zero. k is the only parameter which in the cited works was varied between 3 and 4. Here we use k = 3.9.

3. Finally, we take the Swiss formula for comparison with (9). As already mentioned, it consists of a polynomial of degree 15 with coefficients C_j [3]

$$T_K(h) = \frac{\sqrt{L^2 - h^2}}{1000}\sum_{i=0}^{15} C_i\left[\frac{100h}{\sqrt{L^2 - h^2}}\right]^i \tag{14}$$

which is valid in a range $|h/L| < 0.37$.

The advantage of the physical model (9) compared to the presented descriptive approaches is obvious: all variables have a meaning and influence the walking time which can be studied experimentally. So it would be easy to measure the power as a function of T for a given gradient by using a treadmill. In addition, a mathematical structure has been generated which can be used directly for regression methods.

In the following Figure 3, the three walking time formulas are plotted as a function of the gradient and compared with T(h,l) from equation (9).

The Naismith curve (11), which is linear in the relevant range, kind of averages the convex curves and provides a rough approximation. The walking time from (9) coincides with that of DHN (13), if one chooses P_{opt} = 150W, a = 4.5, and λ = 4/3 sec^{-1}. Choosing P_{opt} = 120W with the same a and λ, equation (9) can reproduce the Swiss formula (14).

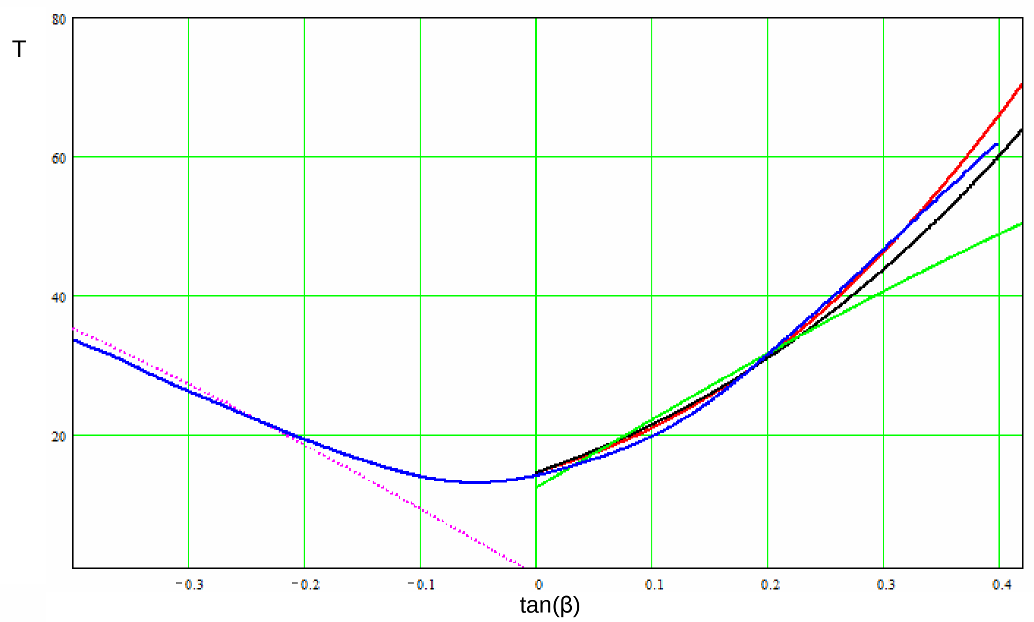

Fig.3: Walking times T in minutes for the different approaches as a function of the gradient for a distance L=1000m. Eq. (9) (red), eq. (14) (blue), eq. (13) (black), eq. (11) (green) and eq. (15) (magenta dots). For eq. (9) P_{opt} = 120W was chosen to obtain a match with the Swiss formula.

The walking time T does not only depend on the gradient but is determined by P_{opt}/m and F_{max}/m. In Figure 4 this strong dependence between the walking time and the power P_{opt} is illustrated.

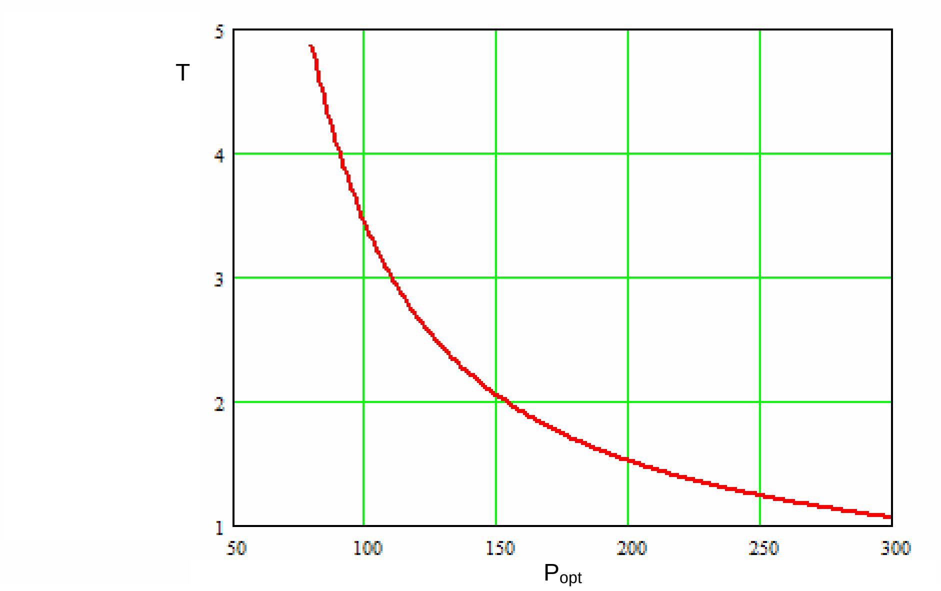

Fig.4: Walking time T of eq. (9) in hours as a function P_{opt}[W].
With h=1000m and L=4000m a typical situation was chosen.

In practice, you can estimate your own individual walking time T_{ind} by means of (9) and the reference point T(P_{opt}=120W) with the standard power P_{st} = 120W from the Swiss formula (therefore valid for Swiss hiking trails). Provided you know your own power P_{ind}, T_{ind} is given by $T_{ind} \approx \frac{P_{st}}{P_{ind}} T_{st}$. For other regions one has an analogous relationship with other reference points.

Walking downhill

Already a very simple approach can explain one part of the downhill movement. To maintain a constant velocity when walking downhill, a braking force mgsin(β) must be applied. For this purpose the power P_{opt} is available, so that $P_{opt} = -mg\dot{h}$.

Solved for the walking time, we get

$$T = \frac{mg|h|}{P_{opt}} \tag{15}$$

The mechanical efficiency for the eccentric braking movement has been set 1. It is remarkable that this simple reasoning can explain the part of the Swiss formula for negative gradients larger than -20% (see Figure 3).

The transition region between T(h=0) and T(h/L≈-0.2) will not be analyzed in more detail, because in this work we are mainly interested in the uphill movement. One thing, however, is certain: the internal friction for the horizontal movement gradually disappears in the transition region, because the hip no longer has to lift against gravity, and thus eventually equation (15) alone determines the downhill walking times.

Energy required and power components

The relation $P = \eta P_{opt}$ can be associated with the required total energy (including heat) for the walking distance, because we have $E_{mech} = T\eta P_{opt} = \eta_1 E$. If the two efficiencies η and η1 are proportional to each other, then also the total energy consumption E is proportional to the walking time T:

$$T \propto E \tag{16}$$

That conclusion is supported by the very good agreement with a measured curve for the total energy consumption (Llobera and Sluckin [11]), as can be seen in Figure 5 below. For comparison, the energy consumption for the efficiency η = 1 is shown as well which is significantly lower for large gradients. This strong increase in energy consumption for steeper gradients eventually leads to the yet to be discussed zigzag transition.

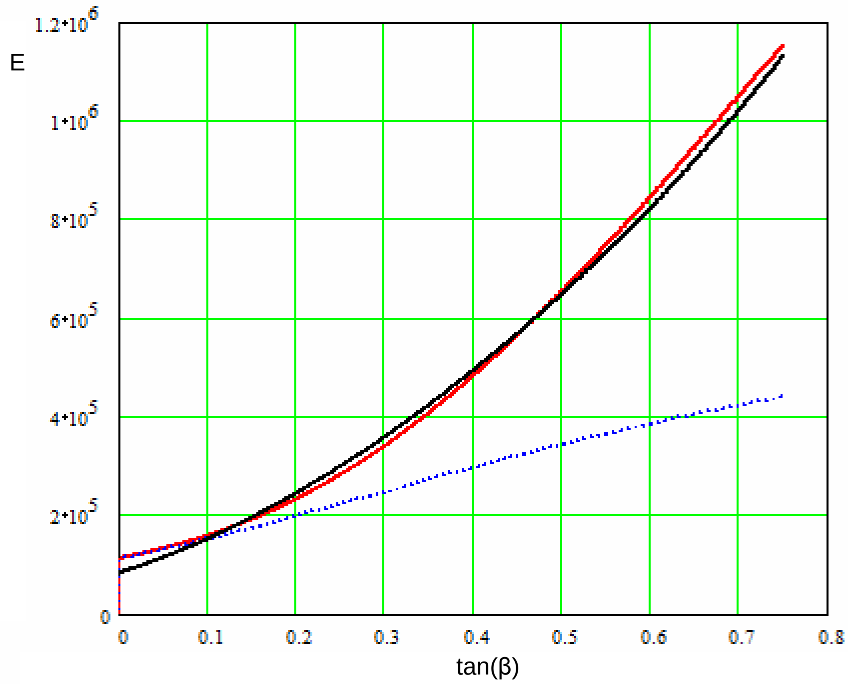

Fig.5: Energy consumption E [J] when walking uphill as a function of the gradient tan(ß) for the model (9,16) (black curve) compared with [11] (red curve) and the energy consumption for an efficiency 1 (dashed blue line).

Figure 6 shows the power components of walking for the ascent and for the horizontal component. For larger gradients the latter is fast approaching zero.

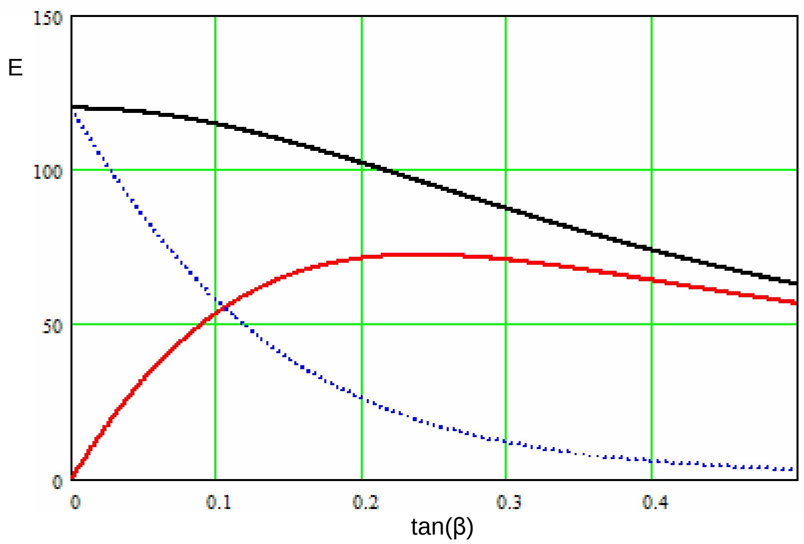

Fig.6: Total power (black curve) made up of the power for the ascent (red curve) and the power for walking horizontally (blue curve) as a function of the gradient.

The zigzag transition

An interesting point is the so-called zigzag transition. Every hiker or ski mountaineer knows that above a critical gradient angle it is better to take a longer way (usually zigzag, see the following Figure 7) to achieve faster uphill walking times. Davey, Hayes and Norman [9], which we already have mentioned, probably were the first who published this. Later, M. Llobera and T.J. Sluckin [11] paid more attention to the zigzag transition.

Fig.7: Zigzag transition on the way to the Stripsenjoch (Wilder Kaiser, Austria)

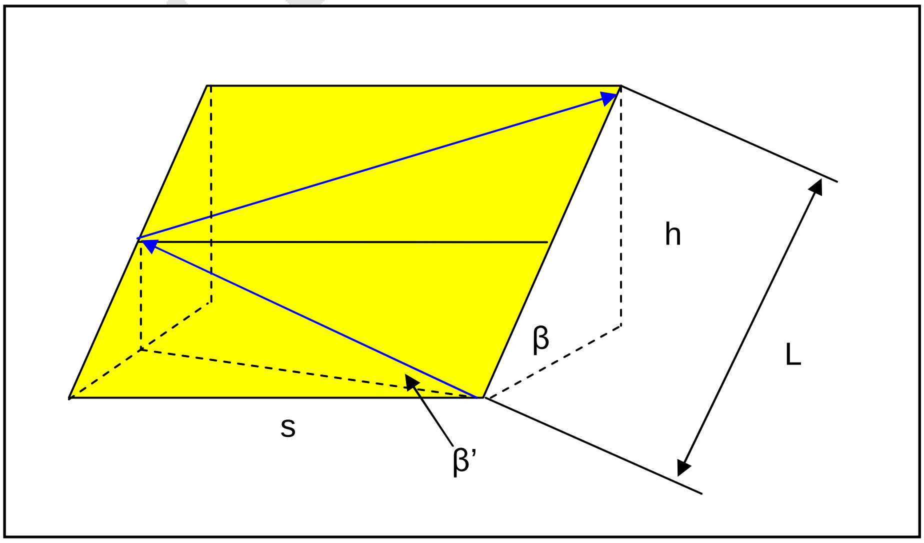

Fig.8 : See text.

One proceeds as follows: zigzag paths with a path length $L' = \sqrt{4s^2 + L^2}$ are allowed where s is the side length of the yellow rectangle in Figure 8. L' is inserted into equation (9). Now we examine whether there exists a s > 0 above a certain $h_c/L = \sin(\beta_c)$ which leads to a shorter walking time than for s = 0.

Figure 9 shows the transition to s > 0 which has been determined numerically.

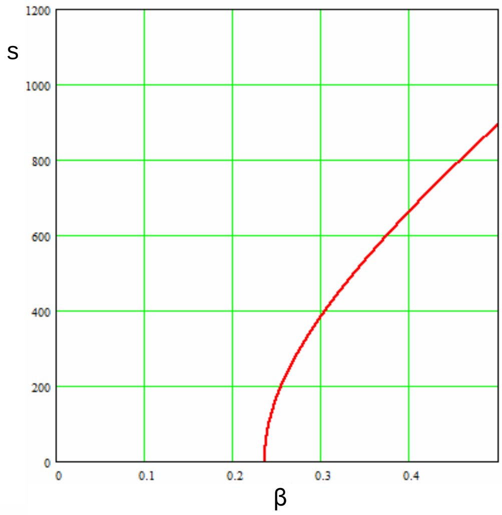

Fig.9: Abrupt onset of a detour, described by s, above a critical gradient angle β_c

Expansion for small s leads to the relation $s \sim (h - h_c)^{\frac{1}{2}}$. This is quite analogous to the Mean-Field Theory for phase transitions (where s plays the role of the order-parameter and h_c the role of the temperature). This is also demonstrated in Figure 10 where the time which has to be minimized (in statistical physics this would be the free energy) is shown for two different gradients. While for smaller gradients the minimum of T is at s = 0, the shortest walking time for larger gradients is achieved by making the "detour" s > 0.

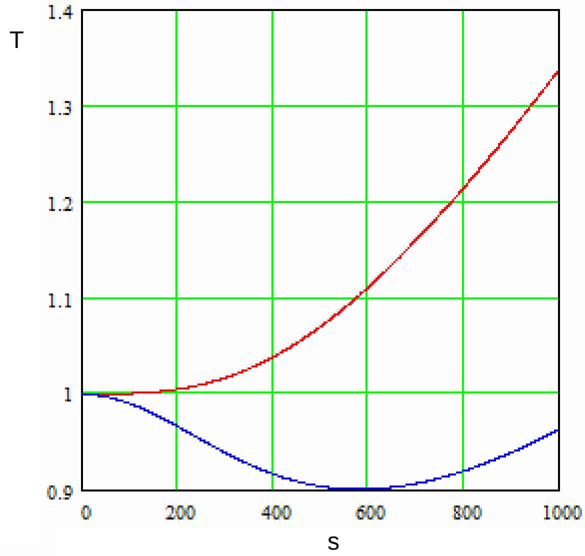

Fig.10: Walking time for 2 different gradients. The red curve is below the critical gradient with the minimum at s = 0, the blue curve is above the critical gradient with s ~ 500m which extends the path of L = 1000m to about L '= 1400m.

The critical angle $\beta_c = \arcsin(h_c/L)$, at which the zigzag transition occurs, can be calculated exactly for the walking time formula (9). It is given by

$$\sin(\beta_c) = \frac{h_c}{L} = \sqrt{\frac{1}{a + mg\sqrt{\frac{a}{\lambda P_{opt}}}}} \tag{17}$$

As expected, the transition disappears for a → 0, because then the efficiency coefficient approaches one and no longer depends on the force. Remarkable is the relatively weak dependence of P_{opt} (see Figure 11). Over a wide range 100W < P_{opt} <300W the zigzag varies only slightly and lies between 13 and 16 degrees.

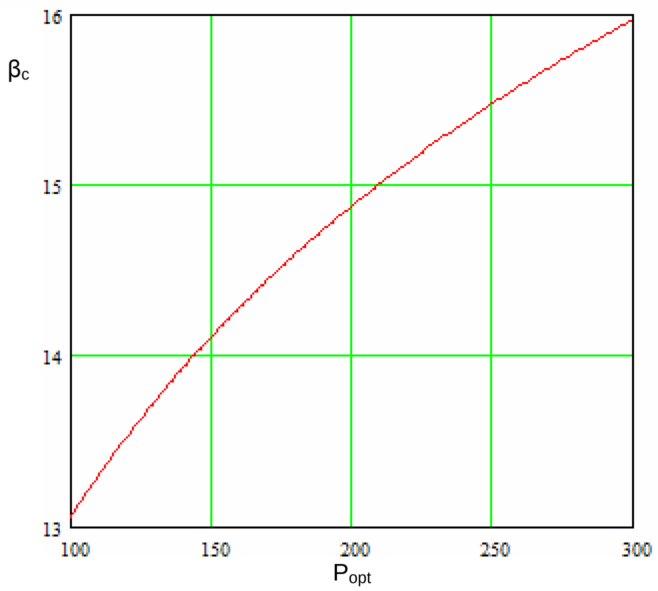

Fig.11: Critical gradient angle (17) in degrees as a function of the power parameter P_{opt}.

In the Swiss formula the zigzag transition occurs at an angle of 13.8° which corresponds to a gradient of 24.6%. The transition of DHN is at 15.4°.

It is important to mention that the Swiss formula has no zigzag transition for walking downhill. Walking times that are quasi linear in gradient, which is also true for the Naismith rule (11), have no zigzag transition.

Conclusion

In this work we derived an equation for the ascent times in mountaineering based on simple principles like the conservation of energy and with a gravity-dependent efficiency coefficient that was derived from Hill's muscle model. Mechanical models à la "inverted pendulum", when generalized to gradients, can not explain the observed walking times. Their gradient contributions to the walking time are too small.

The model reproduces all important physical limiting cases. The equation for the walking time was successfully compared with several descriptive approaches based on measurements. The necessary power parameters are plausible in a range between 100W and 150W. As expected, the walking time T depends not only on the gradient, but is largely determined by the individual characteristics P_{opt}/m and F_{max}/m of the mountain walker (see Figure 4).

The model shows a zigzag transition at a critical gradient angle which is quite insensitive to the available power and lies between 13 and 16 degrees.

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